Topological Degree and its Applications to Elliptic Partial Differential Equations

Jack Thompson

May 30, 2019

Abstract

In this report, we explore topological degree in finite and infinite dimensional spaces with an application to the existence of solutions to a nonlinear, elliptic partial differential equation (PDE). We begin by constructing topological degree for continuously, differentiable maps between smooth, paracompact manifolds, then prove some elementary results, for example, homotopy invariance. We then extend the definition to allow for continuous maps and provide a proof of Brouwer's fixed point theorem. Next, we define the Leray-Schauder degree on Banach spaces and introduce some basic results in the theory of elliptic PDE in order to prove the existence of a smooth solution to a certain mildly nonlinear PDE.

Introduction

Proving the existence of solutions to equations is of fundamental importance for many areas in modern mathematics, especially partial differential equations (PDE). However, to someone who is unfamiliar with the mathematical theory for PDE, such as a physicist or an engineer, the existence of solutions might seem to be of secondary importance.

Consider, for example, the proportion of heat in a medium which is given by the solution of a particular PDE, aptly called the heat equation. Surely, they might say, it is more important to understand how the temperature interacts with other objects in the medium, what is the maximum temperature, or how quickly the heat will spread through the medium? These are all questions regarding the properties of solutions to the heat equation. However, the problem with developing properties of solutions without proving their existence is, if there turns out to be no solutions to the equation, then every property is vacuously true and so we have ultimately achieved nothing. Furthermore, most PDE model something in the real world, so if no solution exists, it shows a shortcoming in our understanding of the process which we are trying to model.

Topological degree is a tool we use to prove the existence of solutions to an equation. The degree of a map in some sense "counts" the number of solutions to an equation, and so if the degree is nonzero, then we have proved the existence of a solution. Proving there is a solution to a given equation has thus been reduced to calculating its degree. However, this task is generally just as difficult as the original problem. The real power of degree is its homotopy invariance property which loosely says that if we can deform one function to another in a continuous way, under certain conditions, then the degree remains unchanged. This allows us to calculate the degree of our equation by deforming it to a much simpler equation for which we can more easily calculate the degree and thereby prove the existence of a solution to our much harder problem.

Throughout this report, we follow Louis Nirenberg's book *Topics in Nonlinear Analysis* [Nir74] very closely. The report is split into two sections: the first is focussed on the theoretical aspects of topological degree and the second on PDE with an example demonstrating how degree can be applied. We begin Section 1 with the construction of degree for continuously differentiable maps between finite dimensional spaces, specifically paracompact, smooth manifolds. We explore some of the properties which were alluded to earlier that are needed in the proofs of the existence of solutions. We then extend the definition of degree to allow for continuous maps which can be done by approximating with continuously differentiable maps for which the degree is already defined. This allows us to prove Brouwer's fixed point theorem. It was in fact Brouwer who originally defined the degree of a map in his celebrated 1912 paper [Bro12] which he then used to provide the first proof of the theorem. We finish Section 1 by introducing the Leray-Schuader degree, which is a further extension of degree to Banach spaces for maps with a particular form. This was originally developed by Jean Leray and Juliusz Schauder in [LS34] and then applied by Leray in [Ler34] to prove the existence of weak solutions to the Navier-Stokes equations, which describe the flow of fluids such as water and air. The construction of the Leray-Schauder degree relies on approximating the map with continuous finite dimensional maps and then using the definition of finite degree.

In Section 2, we apply the results of Section 1 to a particular PDE. Before we can do this, however, we need to introduce some theory for elliptic PDE. We begin by briefly discussing Hölder and Sobolev spaces, which are the common setting for the study of solutions to PDE. We then define and present some basic results regarding elliptic PDE before finishing with an existence result for a smooth solution of a mildly nonlinear, elliptic PDE.

Basic Notation

- $\mathbb{R}^n$: the n-dimensional real number containing points of the form $(x_1, \ldots, x_n)$ where each x_i is a real number.
- $\partial\Omega$: the boundary of Ω.
- $\overline{\Omega}$: the closure of Ω.

- $C^k(\Omega)$: the space of functions from Ω to $\mathbb{R}$ with continuous k-th derivatives. If $k = 1$ we simply write $C(\Omega)$. We allow $k = \infty$ in which case it is the space of function from Ω to $\mathbb{R}$ with infinite derivatives.

- $\operatorname{supp}\phi$: the support of $\phi : X \to Y$ which is defined to be the closure of $\{x \in X : f(x) \neq 0\}$.

- $\operatorname{sgn}\phi$: the sign of ϕ.

- $\operatorname{dist}(x, X) = \inf\{\|x - y\| : y \in X\}$.

1 Topological Degree Theory

1.1 A Motivational Example

Proving the existence of solutions is of great importance in many areas of mathematics. Typically, this involves a map $F : X \to Y$ for suitable spaces X and Y and the question is whether there exists a solution $x \in X$ such that

$$F(x) = 0. \tag{1.1}$$

Often a method to prove such a result is to show that solving (1.1) is equivalent to solving some easier equation. One method of doing this is by using homotopy theory, which is a subject in the theory of algebraic topology.

Definition 1.1. Let X and Y be topological spaces. Two functions $f, g : X \to Y$ are said to be homotopically equivalent or simply homotopic if there exists a continuous function $\phi : [0,1] \times X \to Y$ such that $H(0,\cdot) = f(\cdot)$ and $H(1,\cdot) = g(\cdot)$. We say a function is homotopically trivial if it is homotopic to a constant map.

Remark. 1. Homotopy equivalence forms an equivalence relation.

2. We normally think of t as a parameter. For this reason we will sometimes write the t as a subscript rather than an argument of the function.

Intuitively, if two functions are homotopic then they can be continuously deformed into one another. In the following motivating theorem we see how this can be used to prove existence results. Let B be the closed ball centred at 0 in $\mathbb{R}^n$ and recall $S^{n-1} = \partial B = \{x \in R^n : |x| = 1\}$ is the $(n-1)$-sphere in $\mathbb{R}^n$.

Theorem 1.2. *Suppose $\phi : S^{n-1} \to \mathbb{R}^m - \{0\}$ and define*

$$\psi = \frac{\phi}{|\phi|} : S^{n-1} \to S^{m-1}.$$

Then the following are equivalent.

(i) There exists a solution to $F(x) = 0$ for every continuous extension F of ϕ in B.

(ii) $\psi : S^{n-1} \to S^{m-1}$ is homotopically nontrivial.

Proof. 1. First we prove the negation of (ii) implies the negation of (i). Suppose ψ is homotopically trivial, i.e. there exists continuous $H : [0,1] \times S^{n-1} \to S^{m-1}$ such that $H(0,\cdot) = \psi(\cdot)$ and $H(1,\cdot) \equiv p$ where p is constant in S^{m-1}. Set $M = \max_{S^{n-1}} |\phi(x)|$ (which exists by the extreme value theorem) and define $F : B \to \mathbb{R}^m - \{0\}$ by

$$F(x) = \begin{cases} p, & \text{if } x = 0 \\ p\gamma_1(x), & \text{if } 0 < |x| \leq \frac{1}{2} \\ H\left(2 - 2|x|, \frac{x}{|x|}\right)\gamma_2(x), & \text{if } |x| > \frac{1}{2} \end{cases}$$

where $\gamma_1, \gamma_2 : B - \{0\} \to \mathbb{R}$ are given by

$$\gamma_1(x) = \frac{2|x|}{M+1} \cdot \left| \phi\left(\frac{x}{|x|}\right) \right| + 1 - 2|x|; \text{ and}$$
$$\gamma_2(x) = \left| \phi\left(\frac{x}{|x|}\right) \right| \left(\frac{2 - 2|x|}{M+1} + 2|x| - 1 \right).$$

Now we show F is continuous. Clearly both γ_1 and γ_2 are continuous so it suffices to show F is continuous at $x = 0$ and $|x| = \frac{1}{2}$. Assuming $0 < |x| \leq \frac{1}{2}$, we compute

$$\begin{aligned} |\gamma_1(x)| &= \frac{2|x|}{M+1} \cdot \left| \phi\left(\frac{x}{|x|}\right) \right| + 1 - 2|x| \\ &< 2|x| + 1 - 2|x| = 1; \text{ and} \\ |\gamma_1(x)| &= \left(\frac{2}{M+1} \cdot \left| \phi\left(\frac{x}{|x|}\right) \right| - 2 \right) |x| + 1 \\ &\geq 1 - 2|x| \to 1 \end{aligned}$$

as $|x| \to 0$ and so by the squeeze theorem F is continuous at $x = 0$. If $|x| = \frac{1}{2}$ then

$$F(x) = p\gamma_1(x) = \frac{p}{M+1} |\phi(2x)|.$$

Since H and γ_2 are continuous, for some x such that $|x| = \frac{1}{2}$

$$\begin{aligned} \lim_{|y| \downarrow \frac{1}{2}} F(y) &= H(1, x)\, \gamma_2(x) \\ &= \frac{p}{M+1} |\phi(2x)| = F(x) \end{aligned}$$

and so F is continuous. Furthermore, from our estimates we can see for all $x \in B$, $|F(x)| > 0$ and so there cannot be a solution of $F(x) = 0$ in B.

2. Now we prove the negation of (i) implies the negation of (ii). Suppose there exists a continuous extension F of ϕ such that $F(x) \neq 0$ on B and define $\mu : S^{n-1} \to S^{m-1}$ by

$$\mu(x) = \frac{F(x)}{|F(x)|}.$$

Now let $H(t, x) : [0, 1] \times S^{n-1} \to S^{m-1}$ be given by

$$H(t, x) = \mu((1 - t)x).$$

Since $F(x) \neq 0$ and is continuous, H is well-defined and continuous. Then for $x \in S^{n-1}$ there holds

$$H(1, x) = \mu(x) = \frac{\phi(x)}{|\phi(x)|}.$$

Hence, ψ is homotopic to $H(0, x) \equiv \mu(0)$ i.e. ψ is homotopically trivial. $\square$

The statement of Theorem 1.1 was taken from [Nir74] and the proof was given in a set of lecture notes provided by Joseph Growtowski. Theorem 1.1 tells us if we know the homotopy class of ψ then that is enough to know if any continuous extension of F of ϕ has a solution to $F(x) = 0$. For the case $n = k$, the degree of ψ determines its homotopy class and is homotopically trivial if and only if its degree is zero.

1.2 A Particular Case of Sard's Theorem

In order to define the notion of degree we need to use a special case of Sard's Theorem. Suppose $\phi : X \to Y$ is a mapping, where X and Y are smooth, paracompact manifolds of dimension n and m respectively and $F \in C^1 \cap C^{n-k+1}$. A manifold X is paracompact if it is a Hausdorff space and every open cover has a locally, finite subcover, that is, a subcover such that every point in X intersects finitely many elements in the subcover. However, the condition that X and Y be paracompact is not especially important - all metric spaces are paracompact so it suffices to think of X and Y as smooth, Riemannian manifolds. See [Mun13].

Definition 1.3. 1. A point $q \in X$ is called a regular point if the gradient matrix $D\phi$ has maximal rank i.e. it has rank $\min\{n, k\}$. If $n = k$ this is equivalent to saying the Jacobian is nonzero. If q is not a regular point then it is called a critical point.

2. A point $p \in Y$ is called a critical value if $F^{-1}(\{p\})$ contains a critical point. If p is not a critical value then it is called a regular value.

Theorem 1.4. *(Sard's Theorem) Suppose ϕ is as above. Then the set of critical values has Lebesgue measure zero in Y.*

The proof of Sard's Theorem is difficult and beyond the scope of this report - for the proof see [AR67]. In fact, we only require the case $n = k$.

1.3 Definition of Finite Degree

Let X_0 and Y be smooth, paracompact manifolds of dimension n. Let X be an open subset of X_0 such that $\overline{X}$ is compact, $\phi : X \to Y$ is C^1 and $p \in Y$.

Definition 1.5. A smooth n-form $\mu = f(y)\, dy$ is called admissible for p and ϕ if it has compact support in a 'nice' coordinate patch of p contained in $Y - \phi(\partial X)$ such that

$$\int_Y \mu = 1.$$

Definition 1.6. Let μ be admissible for p and ϕ. We define the degree of ϕ (with respect to X and p) as

$$\deg(\phi, X, p) = \int_X \mu \circ \phi.$$

The degree of ϕ is well-defined in the sense that if μ and ν are admissible for p and ϕ then

$$\int_X \mu \circ \phi = \int_X \nu \circ \phi.$$

The proof is beyond the scope of what we wish to cover but can be found in [Nir74].

1.4 Properties of Finite Degree

Degree is an extremely powerful tool for finding solutions to equations, however, so far the definition seems to offer little to no insight into how this might be the case. In this section we will present some properties of the degree in an effort to demonstrate some of its applications.

Theorem 1.7. *There holds*

$$\deg(\mathrm{Id}, X, p) = \begin{cases} 1, & \text{if } p \in X \\ 0, & \text{if } p \in Y - \phi(\overline{X}) \end{cases}$$

Proof. Let μ be admissible for Id at p. Since p is in the support of μ which is contained in $Y - \mathrm{Id}(\partial X) = Y - \partial X$, $\operatorname{supp} \mu$ is contained in either X or $Y - \phi(\overline{X})$. If $p \in X$ then $\operatorname{supp} \mu \subset X$ so

$$\deg(\phi, X, p) = \int_X \mu \circ \mathrm{Id} = \int_X \mu = 1.$$

Similarly, if $p \in Y - \phi(\overline{X})$ then $\operatorname{supp} \mu \subset Y - \phi(\overline{X})$ so

$$\deg(\phi, X, p) = \int_X \mu = 0.$$

$\square$

Theorem 1.8. *If* $\deg(\phi, X, P) \neq 0$ *then there exists a solution to* $\phi(x) = p$ *in* X*.*

Proof. We prove the contrapositive statement $p \notin \phi(X)$ implies $\deg(\phi, X, p) = 0$. As $p \notin \phi(\overline{X})$ ($p \notin \phi(\partial X)$ by assumption), $\operatorname{supp} \mu \subset Y - \phi(\overline{X})$ for admissible μ so

$$\deg(\phi, X, p) = \int_X \mu \circ \phi = 0.$$

$\square$

Theorem 1.9. *Suppose* μ *is admissible for* ϕ *and* p*. Then for every* $p' \in \operatorname{supp} \mu$*,*

$$\deg(\phi, X, p') = \deg(\phi, X, p)$$

Proof. The proof is trivial since if $p' \in \operatorname{supp} \mu$ then μ is admissible for p' and so

$$\deg(\phi, X, p') = \int_X \mu \circ \phi = \deg(\phi, X, p).$$

$\square$

What this is intuitively saying is that if p and p' are sufficiently close to one another then their degree is the same. This further implies that the degree is constant on connected subsets of $Y - \phi(\partial X)$.

Lemma 1.10. *Suppose p is a regular value of ϕ. Then*

$$\phi^{-1}(p) = \{x_1, \ldots, x_k\}$$

for some $x_1, \ldots, x_k \in X$.

Proof. Observe $\phi^{-1}(p)$ must be discrete, that is for each $x \in \phi^{-1}(p)$, there exists a neighbourhood U of x such that there is no element $x' \neq x$ in $\phi^{-1}(p)$ such that $x' \in U$. This follows from the inverse function theorem since $J_\phi(x) \neq 0$ so there exists a neighbourhood U of x such that $\phi|_U$ is injective and so any other element of $\phi^{-1}(p)$ cannot be in U. Then since $\{p\}$ is closed and $\overline{X}$ is compact, $\phi^{-1}(p)$ is compact and so we must have

$$\phi^{-1}(p) = \{x_1, \ldots, x_k\}$$

for some $x_1, \ldots, x_k \in X$. $\square$

Theorem 1.11. *Suppose p is a regular value of ϕ. Then*

$$\deg(\phi, X, p) = \sum_{j=1}^{m} \operatorname{sgn} J_\phi(x_j)$$

where $\phi^{-1}(p) = \{x_1, \ldots, x_m\}$.

Proof. By the Inverse Function Theorem, let N_j be disjoint open neighbourhoods of each x_j in $\phi^{-1}(p)$ such that $\phi|_{N_j}$ is a homeomorphism onto its image. Define

$$N = \bigcap_{j=1}^{k} \phi(N_j).$$

Since each N_j is open, N is open so we can find an admissible μ for p and ϕ such that $\operatorname{supp} \mu \subset N$. Then

$$\begin{aligned}
\deg(\phi, X, p) &= \int_X (\mu \circ \phi)(x) J_\phi(x)\, dx \\
&= \sum_{j=1}^{k} \int_{N_j} (\mu \circ \phi)(x) J_\phi(x)\, dx \\
&= \sum_{j=1}^{k} \int_{N_j} (\mu \circ \phi)(x)\, |J_\phi(x)| \operatorname{sgn} J_\phi(x)\, dx.
\end{aligned}$$

Since $\phi \in C^1$, the Jacobian is continuous so for all $x \in N_j$, $\operatorname{sgn} J_\phi(x) = \operatorname{sgn} J_\phi(x_j)$which gives

$$\sum_{j=1}^{k} \operatorname{sgn} J_\phi(x_j) \int_{N_j} (\mu \circ \phi)(x) \, |J_\phi(x)| \, dx.$$

Then since ϕ is injective of each N_j, we can make the change of variables $y = \phi(x)$ so we have

$$\sum_{j=1}^{k} \operatorname{sgn} J_\phi(x_j) \int_{\phi(N_j)} \mu = \sum_{j=1}^{k} \operatorname{sgn} J_\phi(x_j)$$

where the second equality comes from $\operatorname{supp} \mu$ is a subset of each $\phi(N_j)$. □

Theorem 1.11 of course implies the degree of a map is an integer if p is a regular value. However, if p is not a regular value - otherwise known as a critical value - then Sard's Theorem tells us there exists a regular value p' close to p and so by Theorem 1.9, the degree is always an integer.

Theorem 1.12. *(Homotopy Invariance) Suppose* $\phi_t : \overline{X} \times [0,1] \to Y$ *is continuous on* $\overline{X} \times [0,1]$ *and is* C^1 *for each* $t \in [0,1]$. *If* $p \notin \phi_t(\partial X)$ *for all* t *then* $\deg(\phi_t, X, p)$ *is independent of* t.

Proof. Let $\tilde{Y} = \{\phi_t(x) : t \in [0,1], x \in \partial X\}$. Then $\tilde{Y}$ is closed and $p \notin \tilde{Y}$ so we can take an admissible μ for ϕ_t and p such that $\operatorname{supp} \mu$ does not intersect $\tilde{Y}$. Then

$$\deg(\phi_t, X, p) = \int_X \mu \circ \phi_t$$

which is continuous. Since the degree is an integer, $\deg(\phi_t, X, p)$ must be constant for all t. □

Theorem 1.13. *Suppose* $\{X_i\}_{i=1}^\infty$ *is a sequence of disjoint, open set in the interior of* X. *If* $p \notin \phi\left(\overline{X} - \bigcup_{i=1}^\infty X_i\right)$ *then* $\deg(\phi, X_i, p) \neq 0$ *for at most a finite number of* X_i*'s. Furthermore,*

$$\deg(\phi, X, p) = \sum_{i=1}^{\infty} \deg(\phi, X_i, p).$$

Proof. Since $p \notin \phi\left(\overline{X} - \bigcup_{i=1}^\infty X_i\right)$ and $\phi\left(\overline{X} - \bigcup_{i=1}^\infty X_i\right)$ is closed we can find an open neighbourhood U of p. Theorem 1.4 tells us the set of critical values has measure zero in Y. If p is a critical value then there exists a regular value p' close to p such that the $\deg(\phi, X, p) = \deg(\phi, X, p')$ so we can assume without loss of generality that p is a regular value. Since $\phi^{-1}(\{p'\})$ is finite, $\phi^{-1}(\{p\})$ only intersects a finite number of X_i's so there can only be a finite number of $\deg(\phi, X_i, p')$ which are not zero. Then Theorem 1.11 implies the result. □

An important corollary of Theorem 1.13 is the following.

Theorem 1.14. *(Excision) If $K \subset \overline{X}$ is closed and $p \notin \phi(K) \cup \phi(\partial X)$ then* $\deg(\phi, X, p) = \deg(\phi, X - K, p)$.

Proof. In Theorem 1.13 let $X_1 = X - K$ and $X_i = \emptyset$ for $i \geq 2$. Then

$$\phi\left(\overline{X} - \bigcup_{i=1}^{\infty} X_i\right) = \phi\left(\overline{X} - (X - K)\right) = \phi(\partial X \cup K) = \phi(\partial X) \cup \phi(K) \not\ni p.$$

Hence,

$$\deg(\phi, X, p) = \sum_{i=1}^{\infty} \deg(\phi, X_i, p) = \deg(\phi, X - K, p).$$

□

The excision property also gives another proof for Theorem 1.8 by setting $K = \overline{X}$ and using the fact that an integral over the empty set is zero.

Theorem 1.15. *Suppose X, Y are manifolds with dimension n and X', Y' are manifolds with dimension m. Let*

$$\phi : X \to Y \qquad \phi' : X' \to Y'$$

be C^1 and suppose the degree of ϕ and ϕ' at p and p' respectively are defined. Then if $\psi = (\phi, \phi')$

$$\deg\left(\psi, X \times X', (p, p')\right) = \deg(\phi, X, p)\deg(\phi', X', p').$$

Proof. Let μ and μ' be admissible for ϕ at p and ϕ' at p' respectively. Then it follows

$$\operatorname{supp}(\mu \cdot \mu') \subset \operatorname{supp}\mu \cap \operatorname{supp}\mu'$$

so $\mu \cdot \mu'$ is admissible for ψ. Then

$$\deg(\psi, X \times X', (p, p')) = \int_{X \times X'} (\mu \cdot \mu') \circ \psi = \int_{X \times X'} (\mu \circ \phi) \cdot (\mu' \circ \psi') = \deg(\phi, X, p)\deg(\phi', X', p').$$

□

Using induction, Theorem 1.15 can be extended to finite sequences of manifolds with different dimensions. The statements of Defintions 1.3, 1.5, 1.6; and Theorems 1.4, 1.8-1.15 along with their proofs were taken directly from [Nir74].

1.5 Miscellaneous Results in Degree Theory

Here we present some useful results that follow almost directly from the properties of degree. We wish to extend our definition of degree to include maps that are continuous. The definition is motivated by the following theorem.

Theorem 1.16. *Suppose $\{\phi_n\}_{n=1}^{\infty}$ is a sequence of C^1 maps which converge uniformly to a continuous map ϕ in $\overline{X}$ and $p \notin \phi(\partial X)$. Then there exists $N \in \mathbb{N}$ such that if $n > N$ then* $\deg(\phi_n, X, p)$ *is independent of n.*

Proof. Since ϕ_n converges uniformly to ϕ there exists some $N \in \mathbb{N}$ such that $n > N$ implies for all $x \in \overline{X}$, $|\phi_n(x) - \phi(x)| < \operatorname{dist}(p, \phi(\partial X))$. Suppose $n > N$ and define for $t \in [0, 1]$

$$\phi_t(x) = t\phi_n(x) + (1-t)\phi_{n+1}(x).$$

If ϕ_t satisfies the condition of Theorem 1.12 then we are done. Clearly, ϕ_t is continuous on $\overline{X} \times [0, 1]$ and is C^1 for each t since by assumption each ϕ_n is C^1. If $x \in \partial X$ then by the triangle inequality

$$|\phi_t(x) - \phi(x)| = t\,|\phi_n(x) - \phi(x)| + (1-t)\,|\phi_{n+1}(x) - \phi(x)| < \operatorname{dist}(p, \phi(\partial X)).$$

Hence, $p \notin \phi_t(\partial X)$, since the converse would mean the above inequality implies there exists $x \in \partial X$ such that $|p - \phi(x)| < \operatorname{dist}(p, \phi(\partial X))$ which is a contradiction. Thus, $\deg(\phi_n, X, p) = \deg(\phi_{n+1}, X, p)$. □

Definition 1.17. Suppose $\{\phi_n\}_{n=1}^{\infty}$ is a sequence of C^1 maps which converge uniformly to a continuous map ϕ in $\overline{X}$ and $p \notin \phi(\partial X)$. Then we define the degree of ϕ at p to be

$$\deg(\phi, X, p) = \lim_{n\to\infty} \deg(\phi_n, X, p).$$

Theorem 1.18. *Definition 1.17 is well-defined*

Proof. Suppose $\{\phi_n\}_{n=1}^{\infty}$ and $\{\psi_n\}_{n=1}^{\infty}$ are sequences of C^1 maps which both converge uniformly to ϕ in $\overline{X}$. Then there exists $N \in \mathbb{N}$ such that $n > N$ implies for all $x \in \overline{X}$, $|\phi_n(x) - \phi(x)| < \operatorname{dist}(p, \phi(\partial X))$ and $|\psi_n(x) - \phi(x)| < \operatorname{dist}(p, \phi(\partial X))$. Suppose $n > N$ and define for $t \in [0, 1]$

$$\phi_t(x) = t\phi_n(x) + (1-t)\psi_n(x).$$

Then by the same arguments in the proof of Theorem 1.16, ϕ_t satisfies the conditions of Theorem 1.12 and so $\deg(\phi_n, X, p) = \deg(\psi_n, X, p)$. □

The statements of Definition 1.17 and Theorem 1.16 are taken from [Nir74]. The proof of Theorem 1.16 was motivated by a hint in [Nir74]. From now on we will assume ϕ is continuous. We must also check for each ϕ there exists a sequence of C^1 maps which converge uniformly to ϕ. In the general setting of manifolds this is an involved process so we will limit our attention to the case $Y = \mathbb{R}^n$. We do this using mollifiers.

Definition 1.19. 1. The standard mollifier $\eta : \mathbb{R}^n \to \mathbb{R}$ is defined to be

$$\eta(x) = \begin{cases} C\exp\left(\dfrac{1}{|x|^2 - 1}\right), & \text{if } |x| < 1 \\ 0, & \text{if } |x| \geq 1 \end{cases}$$

where C is a constant chosen so that $\int_{\mathbb{R}^n} \eta(x)\,dx = 1$.

2. For each $\epsilon > 0$ let

$$\eta_\epsilon(x) = \frac{1}{\epsilon^n}\eta\left(\frac{x}{\epsilon}\right).$$

By a direct computation, it can be shown that η and consequently η_ϵ are smooth. Suppose Ω is an open, bounded set in $\mathbb{R}^n$.

Definition 1.20. Let $u : \Omega \to \mathbb{R}$ be locally integrable, that is, u is integrable on every compact subset of Ω. Define $\Omega_\epsilon = \{x \in \Omega : \text{dist}(x, \partial\Omega) > \epsilon\}$. Then the mollification of u is given by

$$u^\epsilon = \eta_\epsilon * u$$

for each $x \in \Omega_\epsilon$.

Theorem 1.21. *(Properties of mollifiers)*
(i) $\phi^\epsilon \in C^\infty(\Omega)$.
(ii) As $\epsilon \to 0$, $\phi^\epsilon \to u$ *almost everywhere.*
(iii) If $\phi \in C(\Omega)$ *then* $\phi^\epsilon \to u$ *uniformly on compact subsets of* Ω.

The proof of Theorem 1.21 can be found in Appendix §C.4 of [Eva10]. The important properties for our purposes are (i) and (iii). Note we haven't quite proven the existence of a sequence as in Definition 1.17 since the mollification of ϕ only converges to ϕ on compact subsets of X. However, if $U \subset\subset X$ (i.e. $U \subset \overline{U} \subset X$ and $\overline{U}$ is compact in X) such that $p \in phi(U)$ then $K = \overline{X} - U$ satisfies the conditions of Theorem 1.14 and so

$$\deg(\phi, X, p) = \deg(\phi, X - K, p) = \deg(\phi, U, p).$$

Hence, the degree is well-defined in $\mathbb{R}^n$ for continuous functions ϕ.

We conclude this section with one of the most famous results in finite degree theory: Brouwer's Fixed Point Theorem.

Theorem 1.22. *(Brouwer Fixed Point Theorem) Suppose* K *is a closed, bounded, convex subset of* $\mathbb{R}^n$ *and* $f : K \to K$ *is continuous. Then* f *has a fixed point.*

Proof. 1. It is sufficient to prove the result for $K = \overline{B}$ where B is the open unit ball centred at 0 in $\mathbb{R}^n$. This is because K is homeomorphic to $\overline{B}$. To see this pick an arbitrary point x_0 in the interior of K and denote the point at which the straight line through x and x_0 intersects ∂K by $\tilde{x}$. Then define $h : K \to \overline{B}$ by

$$h(x) = \frac{x - x_0}{|\tilde{x} - x_0|}.$$

It can be shown, since K is convex, that h is a homeomorphism. Then let $\psi = h \circ f \circ h^{-1} : \overline{B} \to \overline{B}$. Then if ψ has a fixed point, say x, then $h^{-1}(x)$ will be a fixed point of f.

2. Let $\phi(x) = x - f(x)$. If $0 \in \phi(\partial B)$ then we are done so suppose $0 \notin \phi(\partial B)$. Let $\phi_t(x) = x - tf(x)$ for $0 \le t \le 1$. Clearly, since f is continuous, ϕ_t is continuous on $[0,1] \times \overline{B}$. Furthermore, if $0 \le t < 1$ then $tf(x) \in B$ since $|tf(x)| \le t < 1$ and so if $x \in \partial B$ then

$$|\phi_t(x)| = |x - tf(x)| \ge 1 - |tf(x)| > 0.$$

Hence, $0 \notin \phi_t(\partial B)$ for all $t \in [0,1]$. Then it follows from Theorem 1.12

$$\deg(\phi, \overline{B}, 0) = \deg(\mathrm{Id}, \overline{B}, 0) = 1$$

and so f has a fixed point. □

The first part of the proof of Theorem 1.22 comes from [Llo78].

1.6 Definition of Leray-Schauder Degree

W now wish to extend the results of Section 1.6 to Banach spaces. We want to do this in such a way that the properties of finite dimensional degree carry over to the infinite dimensional case. In particular, we would hope that this new degree would have the following properties:

(i) $\deg(\mathrm{Id}, X, p) = 1$ if $p \in X$;
(ii) $\deg(\phi, X, p) \neq 0$ implies there exists a solution to $\phi(x) = 0$; and
(iii) If ϕ_t is a homotopy such that $p \notin \phi(\partial X)$ for all $t \in [0,1]$ then $\deg(\phi_t, X, p)$ is independent of t.

However, it is not possible, in general, to define a degree that satisfies (i)-(iii) for every $\phi \in C$. Consider the following example.

Example 1.23. Let $X = \ell^2$, B be the open ball in ℓ^2 and $f : \overline{B} \to \overline{B}$ defined by $x \mapsto \big(\sqrt{1 - \|x\|^2}, x_1, x_2, \dots\big)$ where $\|x\| = \sum_{i=1}^{\infty} x_i^2$. Since f is continuous, if Brouwer's fixed point theorem held in ℓ^2 then f must have a fixed point but it does not. If x is a fixed point then $\|x\| = 1$ since $\|f(x)\| = 1$. However, then $\big(\sqrt{1 - \|x\|^2}, x_1, x_2, \dots\big) = \big(0, x_1, x_2, \dots\big) = x$ which implies $0 = x_1 = x_2 = \cdots$. Hence, $\|x\| = 0$ which is a contradiction.

Example 1.23 shows Brouwer's fixed point theorem does not hold in Banach spaces. But Brouwer's fixed point theorem followed from properties (i)-(iii) so these certainly cannot hold in general. It was proved, however, by J. Leray and J. Schauder in 1934 (see [LS34]) that we can extend the degree of maps of the form $\phi = \mathrm{Id} - K$ where K is a compact map. From now on let X denote some Banach space.

Definition 1.24. Let X and Y be Banach spaces. A continuous map $f : X \to Y$ is called compact if for every closed, bounded set $\Omega \subset X$, $\overline{f(\Omega)}$ is compact in Y.

The reason we may define the degree of $\phi = \mathrm{Id} - K$ is due to the following theorem.

Theorem 1.25. *Suppose $\Omega \subset X$ is closed and bounded. If $f : \Omega \to X$ is compact then f is the uniform limit of a sequence of finite dimension maps, that is maps whose image are contained in a finite dimensional subspace.*

Proof. 1. Suppose f is compact and fix $\epsilon > 0$. Then, by definition, $\overline{f(\Omega)}$ is compact so there exists a family of open balls $\{B_i\}_{i=1}^{j(\epsilon)}$ with radius ϵ and centres $\{x_i\}_{i=1}^{j(\epsilon)}$. Let $\{\psi\}_{i=1}^{j(\epsilon)}$ be a partition of unity subordinate to $\{B_i\}_{i=1}^{j(\epsilon)}$. That is, there exists a family of continuous functions $\{\psi\}_{i=1}^{j(\epsilon)}$ from X to $\mathbb{R}$ such that for each i,

(i) $\operatorname{im} \psi_i \subset [0, 1]$;
(ii) $\operatorname{supp} \psi_i \subset B_i$; and
(iii) $\sum_{i=1}^{j(\epsilon)} \psi_i(x) = 1$ for all $x \in \Omega$.

The existence of a partition of unity is due to the fact that Banach spaces are paracompact and paracompact spaces admit partitions of unity, see [Mun13]. Let

$$f_\epsilon(x) = \sum_{i=1}^{j(\epsilon)} \psi_i(f(x))x_i.$$

Clearly from this definition f_ϵ maps into the span of $\{x_i\}_{i=1}^{j(\epsilon)}$, denoted as N_ϵ, which is finite dimensional. Consider

$$\begin{aligned} \|f_\epsilon(x) - f(x)\| &= \left\| \sum_{i=1}^{\epsilon} [\psi_i(f(x))x_i - \psi_i(f(x))f(x)] \right\| \\ &\leq \sum_{i=1}^{j(\epsilon)} \psi_i(f(x)) \|x_i - f(x)\|. \end{aligned}$$

If some $\psi_i(f(x)) > 0$ then $f(x) \in \operatorname{supp} \psi_i \subset B_i$ so $\|x_i - f(x)\| < \epsilon$ and hence for all $x \in \Omega$, $\|f_\epsilon(x) - f(x)\| < \epsilon$.

$\square$

Remark. 1. The reverse statement of Theorem 1.25 is also true but is unnecessary in the construction of Leray-Schauder degree.

2. Theorem 1.25 does not prove that a Banach space will always have the approximation property which was proven in [Dug51] to be false.

3. An n dimensional linear spaces N is an n-manifold with a single coordinate chart which maps elements in $(\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$ to $\sum_{i=1}^{n} \lambda x_i$ for some basis $\{x_i\}_{i=1}^{n}$ of V.

Now we wish to define the Leray-Schauder degree. Suppose Ω is an open, bounded subset of a Banach space X. Let $\phi : \overline{\Omega} \to X$ be of the form $\phi = \mathrm{Id} - K$ where K is a compact map and $p \notin \phi(\partial\Omega)$.

Lemma 1.26. *Suppose ϕ is as above. Then ϕ is a closed map, that is, ϕ maps closed sets to closed sets.*

Proof. Suppose S is a closed subset of $\overline{\Omega}$ and x_m is a sequence in S such that $\phi(x_m)$ converges to y. Then

$$\phi(x_m) = x_m - Kx_m \to y$$

but K is compact so there exists a subsequence $\{Kx_{m_j}\}$ such that $Kx_{m_j} \to z$. It follows

$$x_{m_j} \to y + z = x$$

but S is closed so $x \in S$. Then by continuity of ϕ (compact maps are continuous),

$$y = x - Kx \in \phi(S).$$

Hence, $\phi(S)$ is closed. $\square$

Now since $\partial\Omega$ is closed, Lemma 1.26 implies $\phi(\partial\Omega)$ is closed so there exists some $\epsilon > 0$ such that $2\epsilon < \operatorname{dist}(p, \phi(\partial\Omega))$. Suppose $\{K_\epsilon\}$ is a sequence of functions that converge uniformly to K as in Theorem 1.25 and write N_ϵ for the finite dimensional space that contains p and the image of K_ϵ.

Definition 1.27. Suppose $\Omega \subset X$, $\phi : \Omega \to X$ be of the form $\phi = \mathrm{Id} - K$ for some compact map $K : \Omega \to X$ and $p \notin \phi(\partial\Omega)$. Let $\phi_\epsilon = x - K_\epsilon$ where K_ϵ is as above. The Leray-Schauder degree is defined to be

$$\deg(\phi, \Omega, p) = \deg(\phi_\epsilon, N_\epsilon \cap \Omega, p).$$

We are yet to check the Leray-Schauder degree is well-defined. This is a consequence of the following theorem.

Theorem 1.28. $\deg(\phi_\epsilon, N_\epsilon \cap \Omega, p)$ *is constant for* $0 < \epsilon < \operatorname{dist}(p, \phi_\epsilon(\partial X))$

Proof. Suppose $0 < \epsilon, \eta < \operatorname{dist}(p, \phi_\epsilon(\partial X))$ and let

$$h_t(x) = t\phi_\epsilon(x) + (1 - t)\phi_\eta(x).$$

Then for $x \in \partial\Omega$,

$$\|h_t(x) - p\| \geq \|\phi(x) - p\| - \|h_t(x) - \phi(x)\| > 0$$

since $p \notin \phi(\partial\Omega)$. Hence, by Theorem 1.12,

$$\deg(\phi_\epsilon, N_\epsilon \cap \Omega, p) = \deg(\phi_\eta, N_\eta \cap \Omega, p).$$

$\square$

Perhaps unsurprisingly, almost all of the properties of Brouwer degree transfer analogously to the Leray-Schauder degree. In particular, we get the analogue of the Brouwer fixed point theorem in Banach spaces: the Schauder fixed point theorem.

Theorem 1.29. *(Schauder Fixed Point Theorem) Suppose Ω is a closed, bounded, convex subset of X and $f : \Omega \to \Omega$ is compact. Then f has a fixed point.*

Proof. Let $\{f_\epsilon\}$ be a sequence of functions that converge to f as in Theorem 1.25. Since Ω is convex each f_ϵ maps into $\Omega \cap N_\epsilon$ and so $f_\epsilon : \Omega \cap N_\epsilon \to \Omega \cap N_\epsilon$. Hence, by Brouwer's fixed point theorem, there exists a fixed point $x_\epsilon \in \Omega \cap N_\epsilon$. Then, since the image of each f_ϵ is contained in $\overline{f(\Omega)}$ and f is a compact map, there exists a convergent subsequence $\{f_{\epsilon_j}\}_{j=1}^{\infty}$ that converges to say $x \in \Omega$. Then, since f_ϵ converges uniformly to f, there exists $N \in \mathbb{N}$ such that $j > N$ implies

$$\|x_{\epsilon_j} - f(x_{\epsilon_j}\| = \|f_{\epsilon_j}(x_{m_j}) - f(x_{\epsilon_j}\| < \epsilon.$$

Hence, $f(x_{\epsilon_j})$ converges to x and hence $f(x) = x$. $\square$

All theorems, definitions and examples along with their proofs (where relevant) were taken from [Nir74].

2 Partial Differential Equations

2.1 Hölder and Sobolev spaces

It is well known that $C^k(\Omega)$ is not a good space to study the solutions of partial differential equations - even though $\Delta : C^{k+2}(\Omega) \to C^k(\Omega)$ is continuous it is not surjective. Hölder spaces are the first step towards fixing this problem. Suppose $0 < \gamma \leq 1$. A function $u : \Omega \to \mathbb{R}$ is γ - Hölder continuous if there exists a constant $C > 0$ such that for all $x, y \in \Omega$

$$|u(x) - u(y)| \leq C\,|x - y|^\gamma\,.$$

If $\gamma = 1$ then we call u Lipschitz continuous. The γ-th Hölder seminorm is defined to be

$$[u]_{C^{0,\gamma}(\bar{\Omega})} = \sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{|u(x) - u(y)|}{|x - y|^\gamma}.$$

This is not quite a norm (as suggested by the name) since $u \equiv 1$, for example, has γ-th Hölder seminorm 0.

Notation. A multi-index is an n-tuple $\alpha = (\alpha_1, \ldots, \alpha_n)$ such that each α_j is a nonnegative integer. The order of α, denoted $|\alpha|$, is given by

$$\sum_{j=1}^n \alpha_j.$$

Then we write

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$$

and for $x \in \mathbb{R}^n$

$$x^\alpha = x_1^{\alpha_1} \cdot x_n^{\alpha_n}.$$

This notation drastically simplifies expressions such as the multivariate Taylor series, the multinomial theorem and the multivariate Leibniz differentiation rule.

Definition 2.1. The γ-th Hölder space

$$C^{k,\gamma}(\bar{\Omega})$$

is the set of functions such that the norm

$$\|u\|_{C^{k,\gamma}(\bar{\Omega})} = \sum_{|\alpha|\leq k} \sup_\Omega |D^\alpha u| + \sum_{|\alpha|=k} [D^\alpha u]_{C^{0,\gamma}(\bar{\Omega})}$$

is finite.

Even better spaces to study solutions of partial differential equations are the Sobolev spaces.

Definition 2.2. The Sobolev space $W^{k,p}(\Omega)$, $k \geq 0$ is an integer and $1 \leq p \leq \infty$, is the closure of $C^{\infty}(\Omega)$ with the norm

$$\|u\|_{W^{k,p}(\Omega)} = \begin{cases} \left(\displaystyle\sum_{|\alpha| \leq k} \int_{\Omega} |D^{\alpha} u|^p \, dx \right)^{\frac{1}{p}}, & \text{if } 1 \leq p < \infty \\ \displaystyle\sum_{|\alpha| \leq k} \operatorname*{ess\,sup}_{\Omega} |D^{\alpha} u|, & \text{if } p = \infty \end{cases}$$

Remark. Most introductory textbooks on Sobolev spaces use an equivalent definition with weak derivative (see, for example, [Eva10]). However, we opted for this definition because the motivation and formulation of Sobolev spaces using weak derivatives is lengthy and unnecessary for understanding the content in this report.

The following is an important theorem which describes a relationship between Sobolev spaces and Hölder spaces.

Theorem 2.3. *Suppose Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary. If $u \in W^{m,p}(\Omega)$ with m a positive integer and $1 \leq p < \infty$. Then ,for every integer $j \in [0, m)$ such that*

$$\mu = m - \frac{n}{p} - j \in (0, 1),$$

$u \in C^{j,\mu}(\Omega)$.

For the proof see [Ada75]. Theorem 2.3 is the second part of a more general statement which can be found in [Eva10]. For a more comprehensive introduction to Hölder and Sobolev spaces see [Eva10].

2.2 Elliptic Partial Differential Equations

In this section we explore applications of the previously established theory of degree to the theory of partial differential equations, in particular, the existence of smooth solutions to boundary valued problems.

Let Ω be an open, bounded subset of $\mathbb{R}^n$ with smooth boundary. An operator $L : X \to Y$, for suitable spaces X and Y, is an order m linear partial differential operator with smooth coefficients $a_{\alpha} : \overline{\Omega} \to \mathbb{R}$ if it is given by

$$Lu = \sum_{|\alpha| \leq m} a_{\alpha}(x) D^{\alpha} u.$$

Such an operator is called elliptic if, for every $x \in \overline{\Omega}$ and $\xi \in \mathbb{R}^n - \{0\}$, the associated polynomial

$$P(x, \xi) = \sum_{|\alpha| = m} a_{\alpha}(x) \xi^{\alpha}$$

is nonzero. The Laplacian

$$\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$$

is the quintessential example of a second order elliptic operator and has associated polynomial

$$P(x,\xi) = |\xi|^2 .$$

For each elliptic operator there is a corresponding boundary valued problem

$$\begin{cases} Lu = f & \text{in } \Omega \\ Bu = g & \text{on } \partial\Omega \end{cases} \tag{2.1}$$

where f, g are given functions and B is a differential operator of order less than m called the boundary operator. Common boundary operators are $B = \mathrm{Id}$ and $B = \frac{\partial}{\partial \nu}$ (here ν is the outward pointing unit normal on $\partial\Omega$) which correspond to Dirichlet and Neumann boundary conditions respectively. We would like to say something about whether (2.1) is well-posed, in some sense. We would love this to mean there exists a solution to (2.1) and it is unique, however, this is often too strong a condition so we must relax our definition.

Definition 2.4. We say (2.1) is well-posed if

1. $\ker L \subset C^\infty$ and is finite dimensional

2. L is continuous and has closed range in Y with finite codimension.

Then the index of L is defined to be $\operatorname{ind} L = \dim\ker L - \operatorname{codim}\operatorname{im} L$.

The following are some important results in the theory of elliptic operators that we will require later in the report.

Theorem 2.5. *The operator* $L : \{u \in W^{k+m,p}(\Omega) : u = 0 \text{ on } \partial\Omega\} \to W^{k,p}(\Omega)$ *is well-defined.*

Theorem 2.6. *Suppose* $u \in W^{k+m,p}(\Omega)$. *Then*

$$\|u\|_{W^{k+m,p}(\Omega} \leq C\|Lu\|_{W^{k,p}(\Omega)} + \|u\|_{L^2(\Omega)}.$$

Furthermore, if $\ker L = \{0\}$ *then*

$$\|u\|_{W^{k+m,p}(\Omega} \leq C\|Lu\|_{W^{k,p}(\Omega)}.$$

Theorem 2.7. *If* $\mu + k > \mu' + k'$ *for* $\mu, \mu' \in (0,1)$ *and* k, k' *positive integers then the open ball with radius* $R > 0$

$$\{u \in C^{k,\mu}(\Omega) : \|u\|_{C^{k,\mu}(\Omega)} < R\}$$

is compact in $C^{k',\mu'}(\Omega)$.

All content on elliptic partial differential equations including the Definition 2.4 and Theorems 2.6-2.7 were taken from [Nir74]. For the proofs see [Fri69] and for for more general statements see [H$\ddot{6}$9].

2.3 A Mildly Nonlinear Elliptic Equation

In this section, we are going to show an application of degree theory by proving the existence of solutions to a mildly nonlinear elliptic equation. Suppose $\Omega \subset \mathbb{R}^n$ is bounded with smooth boundary and L is elliptic. We will be considering the following boundary value problem.

$$\begin{cases} Lu = g(x, u, D^\alpha u), & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega \end{cases} \tag{2.2}$$

where g is smooth with respect to x and u and satisfies the estimate

$$|g(x, u, D^\alpha u)| \leq M \left(1 + \sum_{|\alpha|<m} |D^\alpha u|\right)^\gamma$$

for some $M > 0$ and $0 < \gamma < 1$.

Theorem 2.8. *Suppose* $\ker L = \{0\}$ *and* $\operatorname{ind} L = 0$*. Then there exists a solution* $u \in C^\infty(\overline{\Omega})$ *which satisfies (2.2).*

Remark. 1. Write $Gu = g(x, u, D^\alpha u)$ and note $\ker L = \{0\}$ and $\operatorname{ind} L = 0$ is equivalent to saying L is invertible. Then, under the assumptions of Theorem 2.8, proving the existence of a solution to (2.2) is equivalent to showing there is a fixed point of the map $L^{-1}Gu$ in $C^\infty(\Omega)$.

2. The conclusion of Theorem 2.8 is valid for (2.2) with boundary condition $Bu = 0$ on $\partial\Omega$ where B is either a coercive, complementing or Lopantinsky-Shapira boundary operator. See [Shm65].

Before we prove Theorem 2.8 we make some a priori estimates.

A Priori Estimates. Suppose $u \in W^{m,p}(\Omega)$ solves (2.2). Then by Theorem 2.6 there holds

$$\|u\|_{W^{m,p}(\Omega)} \leq \|Lu\|_{W^{0,p}(\Omega)} = \|Gu\|_{W^{0,p}(\Omega)} \leq C \left[\int_\Omega \left(1 + \sum_{|\alpha|<m} |D^\alpha u|\right)^{\gamma p} dx\right]^{\frac{1}{p}}.$$

If $\sum_{|\alpha|<m} |D^\alpha u| < 1$ then clearly u is bounded in $W^{m,p}(\Omega)$. If $\sum_{|\alpha|<m} |D^\alpha u| \geq 1$ then

$$\begin{aligned} \left[\int_\Omega \left(1 + \sum_{|\alpha|<m} |D^\alpha u|\right)^{\gamma p} dx\right]^{\frac{1}{p}} &\leq C \left[\int_\Omega \left(\sum_{|\alpha|<m} |D^\alpha u|\right)^{\gamma p} dx\right]^{\frac{1}{p}} \\ &\leq C\, |\Omega|^{1-\gamma} \left[\int_\Omega \left(\sum_{|\alpha|<m} |D^\alpha u|\right)^{p} dx\right]^{\frac{\gamma}{p}} \end{aligned}$$

by Hölder's inequality where $|\Omega|$ denotes the Lebesgue measure of Ω, which is finite since Ω is bounded. Then by Minkowski's inequality this is less than or equal to

$$\begin{aligned} C\left[\sum_{|\alpha|<m}\left(\int_\Omega |D^\alpha u|^p\,dx\right)^{\frac{1}{p}}\right]^\gamma &\le C\left[\sum_{|\alpha|\le m}\left(\int_\Omega |D^\alpha u|^p\,dx\right)^{\frac{1}{p}}\right]^\gamma \\ &\le C\left[\sum_{|\alpha|\le m}\left(\int_\Omega \sum_{|\alpha|\le m}|D^\alpha u|^p\,dx\right)^{\frac{1}{p}}\right]^\gamma \\ &= C\|u\|^\gamma_{W^{m,p}(\Omega)} \end{aligned}$$

Hence, in the case $\sum_{|\alpha|<m}|D^\alpha u| \ge 1$ there holds

$$\|u\|_{W^{m,p}(\Omega)} \le C\|u\|^\gamma_{W^{m,p}(\Omega)}.$$

Dividing through by $\|u\|^\gamma_{W^{m,p}(\Omega)}$, we conclude u is always bounded in $W^{m,p}(\Omega)$.

Now if we assume $p > n$ then $\mu = 1 - \frac{n}{p}$ is in $(0,1)$. Then by the Theorem 2.3 with $j = m-1$

$$\|u\|_{C^{m-1}(\Omega)} \le \|u\|_{C^{m-1,\mu}(\Omega)} \le C\|u\|_{W^{m,p}(\Omega)}$$

and so we conclude u is bounded in $C^{m-1}(\Omega)$.

Proof. 1. Let $X = \{u \in C^{m-1}(\Omega) : Bu = 0\}$. By our a priori estimates we know if a solution to (2.2) exists then it is bounded in X by some constant $C_1 > 0$. Let

$$\Sigma = \{u \in X : \|u\|_{C^{m-1}(\Omega)} \le C_1 + 1\}.$$

Let $\phi(u) = u - L^{-1}Gu$ for $u \in \Sigma$. It is clear $0 \notin \phi(\partial\Sigma)$ since if it were then (2.2) would have a solution such that its norm in $C^{m-1}(\Omega)$ was $C_1 + 1$ which contradicts our a priori estimates. Furthermore, for all $u \in \bar{\Sigma}$, it follows for each $|\alpha| < m$ that $|D^\alpha u| \le C_1 + 1$ and so

$$|(Gu)(x)| \le M\left(1 + \sum_{|\alpha|<m}|D^\alpha u|\right)^\gamma \le C_2.$$

Hence, by Theorem 2.6

$$\|L^{-1}Gu\|_{W^{m,p}(\Omega)} \le C\|Gu\|_{W^{0,p}(\Omega)} \le C_3. \tag{2.3}$$

Setting $p > n$ as before, Theorem 2.3 imply

$$\|L^{-1}Gu\|_{C^{m-1,\mu}(\Omega)} \le C_4$$

for $\mu = 1 - \frac{n}{p}$. Theorem 2.7 then tells us $L^{-1}Gu$ is a compact map from $\bar{\Sigma}$ to $C^{m-1}(\Omega)$ and so $\deg(\phi, \Sigma, 0)$ is well-defined.

Furthermore, if we let $\phi_t(u) = u - tL^{-1}Gu$, $0 \leq t \leq 1$, then $0 \notin \phi_t(\partial\Sigma)$ for if it were then there would exist a u such that $Lu = tGu$ which, by our a priori estimates, implies

$$\|u\|_{C^{m-1}(\Omega)} \leq tC_1 \leq C_1$$

and so contradicts $\|u\|_{C^{m-1}(\Omega)} = C_1 + 1$. It follows, by homotopy invariance,

$$\deg(\phi, \Sigma, 0) = \deg(\mathrm{Id}, \Sigma, 0) = 1$$

and so there exists a solution to (2.2) in X.

2. Now we prove $u \in C^\infty(\overline{\Omega})$. By (2.3)

$$\|u\|_{W^{m,p}(\Omega)} = \|L^{-1}Gu\|_{W^{m,p}(\Omega)} \leq C_5$$

so $u \in W^{k,p}(\Omega)$. Furthermore, $Gu \in W^{1,p}(\Omega)$ since $u \in \Sigma$ and so it follows

$$\|u\|_{W^{m+1,p}(\Omega)} = \|L^{-1}Gu\|_{W^{m+1,p}(\Omega)} \leq C\|Gu\|_{W^{0,p}(\Omega)} \leq C'.$$

Then Theorem 2.3 implies $u \in C^m(\overline{\Omega})$. Continuing in this way we get the result. $\square$

References

[Ada75] Robert Alexander Adams. *Sobolev spaces / Robert A. Adams.* Pure and applied mathematics (Academic) ; v. 65. Academic Press, New York, 1975.

[AR67] Ralph Abraham and Joel Robbin. *Transversal Mappings and Flows.* 1967.

[Bro12] L. E. J. Brouwer. über abbildung von mannigfaltigkeiten. *Mathematische Annalen*, 71(4):598–598, 1912.

[Dug51] J Dugundji. An extension of tietze's theorem. *Pacific J. Math*, 1:353–367, 1951.

[Eva10] Lawrence C. Evans. *Partial differential equations / Lawrence C. Evans.* Graduate studies in mathematics; v. 19. American Mathematical Society, Providence, R.I., 2nd ed.. edition, 2010.

[Fri69] Avner Friedman. *Partial differential equations.* Holt, Rinehart and Winston, New York, 1969.

[Ḧ69] Lars Hörmander. *Linear Partial Differential Operators by Lars Hörmander.* Grundlehren der mathematischen Wissenschaften, A Series of Comprehensive Studies in Mathematics, 116. Springer Berlin Heidelberg : Imprint: Springer, Berlin, Heidelberg, third revised printing.. edition, 1969.

[Ler34] Jean Leray. Sur le mouvement d'un liquide visqueux emplissant l'espace. *Acta Math.*, 63:193–248, 1934.

[Llo78] Noel Glynne Lloyd. *Degree Theory.* Cambridge University Press, Cambridge, 1978.

[LS34] Jean Leray and Jules Schauder. Topologie et équations fonctionnelles. *Annales scientifiques de l'École Normale Supérieure*, 3e série, 51:45–78, 1934.

[Mun13] James Munkres. *Topology: Pearson New International Edition.* Pearson Education Limited, Harlow, United Kingdom, UNITED KINGDOM, 2013.

[Nir74] Louis Nirenberg. *Topics in Nonlinear Functional Analysis*, volume 6. American Mathematical Soc., 1974.

[Shm65] Agmon Shmuel. *Lectures on elliptic boundary value problems / by Shmuel Agmon ; Prepared for publication by B. Frank Jones, with the assistance of George W. Batton.* Van Nostrand mathematical studies ; no.2. Van Nostrand, Princeton, N.J., 1965.