The Leray Projection

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Overview

- **Goal:** Rigourosly define the Leray projection on $\mathbb{R}^n$, $\mathbb{R}^n \setminus \mathbb{Z}^n$ and apply it to the incompressible Navier Stokes equations.

- **Plan:** 1) Some motivation
  2) Leray Systems
     2a) Uniqueness
     2b) Existence

Motivation

Incompressible Navier Stokes equations:

Issue!

$$\begin{cases} u_t + (u\cdot\nabla)u = -\nabla p + \nu\Delta u + F, & \text{in } \mathbb{R}^n\times(0,\infty) \\ \operatorname{div} u = 0, & \text{in } \mathbb{R}^n\times(0,\infty) \end{cases}$$

Why? Formally, take divergence of the first line:

$$\operatorname{div}(u_t) = (\operatorname{div} u)_t = 0, \quad \operatorname{div}(-\nabla p) = -\Delta p$$

$$\operatorname{div}(\Delta u) = \Delta \operatorname{div} u = 0.$$

Motivation

$$\operatorname{div}(u\cdot\nabla)u = \sum_{i=1}^{n}\frac{\partial}{\partial x_i}\sum_{j=1}^{n}u_j\frac{\partial u_i}{\partial x_j}$$

$$= \sum_{i,j=1}^{n}\frac{\partial u_j}{\partial x_i}\frac{\partial u_i}{\partial x_j} + \sum_{j=1}^{n}u_j\frac{\partial}{\partial x_j}\sum_{i=1}^{n}\frac{\partial u_i}{\partial x_i}$$

$$= \sum_{i,j=1}^{n}\frac{\partial u_j}{\partial x_i}\frac{\partial u_i}{\partial x_j} + \underbrace{\sum_{j=1}^{n}u_j\frac{\partial}{\partial x_j}\operatorname{div} u}_{=0}$$

$$= \sum_{i,j=1}^{n}\frac{\partial u_j}{\partial x_i}\frac{\partial u_i}{\partial x_j}.$$

Motivation

Hence,

$$\boxed{-\Delta p = \sum_{i,j=1}^{n} \frac{\partial u_j}{\partial x_i}\frac{\partial u_i}{\partial x_j} - \operatorname{div} F}$$

↳ Poisson equation!

Thus,* if $f(x) := \sum_{i,j=1}^{n} \frac{\partial u_j}{\partial x_i}\frac{\partial u_i}{\partial x_j} - \operatorname{div} F$ then

$$p(x) = \int_{\mathbb{R}^n} \underbrace{\Phi(x-y)}_{\text{Fundamental soln of Laplace's equation}} f(y)\,dy \rightsquigarrow \text{Nonlocal!}$$

* Assuming F, p, u decay rapidly.

Motivation

Question: Can we reformulate (INS) without the pressure term?

Answer: Yes!

Let $u : \mathbb{R}^n \to \mathbb{R}^n$ be smooth and rapidly decaying. The Fourier transform of u is

$$\mathcal{F}\{u\}(y) = \hat{u}(y) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} u(x)\, e^{-i x \cdot y}\, dx$$

Vector

Motivation

The inverse Fourier transform of u:

$$\mathcal{F}^{-1}\{u\}(x) = \check{u}(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} u(y)\, e^{i x \cdot y}\, dy$$

The Leray projection of u:

$$\mathbb{P}(u) = u - \nabla \Delta^{-1} (\operatorname{div} u)$$

which is understood in the sense

Fourier symbol

$$\mathbb{P}(u) = \mathcal{F}^{-1}\{ S(y)\, \hat{u}(y) \} \qquad S(y) = I_n - \frac{y \otimes y}{|y|^2}$$

Motivation

Properties of the Leray projection:

Let $u \in (\mathcal{S}(\mathbb{R}^n))^n$ i.e. smooth and rapidly decaying.

1) $\mathbb{P}(\mathbb{P}(u)) = \mathbb{P}(u)$
2) $\operatorname{div} \mathbb{P}(u) = 0$
3) If $\operatorname{div} u = 0$ then $\mathbb{P}(u) = u$.
4) If $p \in \mathcal{S}(\mathbb{R}^n)$ then $\mathbb{P}(\nabla p) = 0$.
5) If $p \in \mathcal{S}(\mathbb{R}^n)$ then $(\nabla p, \mathbb{P}(u))_{L^2(\mathbb{R}^n)} = 0$.

Proof. 1) – 4) Exercise.

$\square$

Proof of 5) Since the Fourier transform is an isometry from $L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$,

$$(\nabla p, \mathbb{P}(u))_{L^2(\mathbb{R}^n)} = (\mathcal{F}\{\nabla p\}, \mathcal{F}\{\mathbb{P}(u)\})_{L^2(\mathbb{R}^n)}$$

$$= \int_{\mathbb{R}^n} i\,\hat{p}(y) y \cdot \left(I_n - \frac{y \otimes y}{|y|^2} \hat{u}(y) \right) dy$$

$$= \int_{\mathbb{R}^n} i\,\hat{p}(y) y \cdot \hat{u}(y) dy - \int_{\mathbb{R}^n} i\,\hat{p}(y)\, y \cdot \frac{y \otimes y}{|y|^2} \hat{u}(y) dy$$

Since $y \cdot \frac{y \otimes y}{|y|^2} \hat{u} = \sum_{j,k=1}^{n} \frac{y_j^2 y_k}{|y|^2} \hat{u}_k = y \cdot \hat{u}$ it follows

$$(\nabla p, \mathbb{P}(u))_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} i\,\hat{p}(y) y \cdot \hat{u}(y) dy - \int_{\mathbb{R}^n} i\,\hat{p}(y) y \cdot \hat{u}(y) dy = 0$$

$\square$

Motivation

Return to (INS): $\begin{cases} u_t + (u\cdot\nabla)u = -\nabla p + \nu\Delta u + F \\ \operatorname{div} u = 0 \end{cases}$

Apply $\mathbb{P}$:

- $\mathbb{P}(u_t) = \mathbb{P}(u)_t = u_t$
- $\mathbb{P}(\nabla p) = 0$

Write $Au := -\mathbb{P}(\Delta u)$, $B(u,v) = \mathbb{P}((u\cdot\nabla)v)$

Stoke's operator

Bilinear

$\tilde{u} : t \longmapsto (x \longmapsto u(x,t))$

$\tilde{u}$ maps time to a function.

Motivation

Then (INS) becomes:

$$(*) \quad \frac{d\tilde{u}}{dt} + \nu A\tilde{u} + B(\tilde{u}, \tilde{u}) = \mathbb{P}(F)$$

↳ Functional differential equation

Benefits:

- Got rid of p.
- Can use $(*)$ to define notion of weak solution.

A note on the Stokes' operator $Au := -\mathbb{P}(\Delta u)$

Why isn't $A = -\mathbb{P}\Delta u = -\Delta \mathbb{P}(u) = -\Delta u$?

- On $\mathbb{R}^n \setminus \mathbb{Z}^n$ this is true.
- On a bounded domain Ω, this is not always true.
- On $\mathbb{R}^n$???

Leray Systems

<u>Problem</u>: Given a vector field $F:\mathbb{R}^n \to \mathbb{R}^n$ can we write F as the sum of a divergence free vector field and a conservative vector field?

i.e. Given $F:\mathbb{R}^n \to \mathbb{R}^n$ can we find $v:\mathbb{R}^n \to \mathbb{R}^n$ and $p:\mathbb{R}^n \to \mathbb{R}$ such that

$$\begin{cases} F = v + \nabla p \\ \operatorname{div} v = 0 \end{cases} ?$$

Leray Systems

- This is called the Helmholtz decomposition since we are working on $\mathbb{R}^n$.

- If we considered the problem on a bounded domain Ω with "no-slip" condition $u=0$ on $\partial\Omega$ then this is called the Helmholtz-Leray decomposition.

Leray Systems

Formally, take divergence of both sides:

$$\operatorname{div} F = \operatorname{div}(v + \nabla p) = \Delta p$$

Hence, $p = \Delta^{-1} \operatorname{div} F$. It follows

$$v = F - \nabla p = \underbrace{F - \nabla \Delta^{-1} \operatorname{div} F}_{\mathbb{P}(F)}$$

- $\mathbb{P}(F)$ projects F into space of divergence-free v.f.s.
- To rigorously define $\mathbb{P}$, we need existence & uniqueness of Leray systems.

Uniqueness

Leray Systems

- By linearity, it is enough to understand the case $F=0$.

- $F=0 \rightsquigarrow v=-\nabla p$, $\operatorname{div} v=0 \rightsquigarrow \Delta p=0$.

$$\boxed{\text{Uniqueness is always up to a harmonic function}}$$

Suppose F, v, p are smooth.

Suppose $p \in L^q(\mathbb{R}^n)$, $q \in [1,\infty)$

In $\mathbb{R}^n$

mean-value formula

$$p(x) = \fint_{B_R(x)} p(y)\,dy \quad \forall R>0$$

Hölder inequality

$$\leq \frac{1}{|B_R(x)|} \cdot |B_R(x)|^{1-1/q} \|p\|_{L^q(\mathbb{R}^n)}$$

$$\leq C R^{-q/n} \|p\|_{L^q(\mathbb{R}^n)}$$

Send $R \to \infty$, $p(x) = 0$.

Solutions are unique.

Suppose $v \in L^q(\mathbb{R}^n;\mathbb{R}^n)$, $q \in [1,\infty)$

Let u be harmonic w/ $\nabla u \in L^q(\mathbb{R}^n;\mathbb{R}^n)$

$p \to p+u$, $v \to v-u$

is still a solution

∇u is harmonic $\rightsquigarrow$ $\nabla u = 0$

$\hookrightarrow u(x) = b$, $b \in \mathbb{R}$

Solutions are unique up to the addition of a constant.

$\boxed{\text{In } \mathbb{R}^n \setminus \mathbb{Z}^n}$

Suppose F, p, v are periodic.

- $F = 0 \rightsquigarrow p$ is harmonic
  ↳ Liouville's thm $\Rightarrow$ p is a constant

Solutions are unique up to a constant.

Remarks:

1) Since only ∇p appears in (INS), this freedom almost entirely irrelevant to us.

2) We can remove this freedom by taking $\int_{\mathbb{R}^n \setminus \mathbb{Z}^n} p \, dx = 0$.

Note: Suppose only F, v are periodic.

Then we can modify p by a harmonic function u which need not be periodic, but whose gradient ∇u is periodic.

Since ∇u is harmonic,

Liouville's thm $\Rightarrow$ $\nabla u = a$ for some $a \in \mathbb{R}^n$.

$\hookrightarrow$ $u(x) = a \cdot x + b$, $b \in \mathbb{R}^n$

Solutions are unique up to affine function.

Existence

Leray Systems

In $\mathbb{R}^n$

For each $F \in \mathcal{S}(\mathbb{R}^n)^n$, define

$$\mathbb{P}(F) = \mathcal{F}^{-1}\{ S(y)\hat{F}(y)\}, \qquad S(y) = I_n - \frac{y \otimes y}{|y|^2}$$

By our previous calculations

$$v = \mathbb{P}(F), \qquad p = \mathcal{F}\left\{ -i\frac{y\cdot\hat{F}}{|y|^2}\right\} \left(= \Delta^{-1}\operatorname{div} F\right)$$

Now let $s \geq 0$. For each $u \in L^2(\mathbb{R}^n;\mathbb{R}^m)$, define

$$\|u\|_{H^s(\mathbb{R}^n;\mathbb{R}^m)} = \left(\int_{\mathbb{R}^n} (1+|y|^2)^s \, |\hat{u}(y)|^2 \, dy \right)^{1/2}$$

$$= \| (1+|y|^2)^{s/2} \hat{u} \|_{L^2(\mathbb{R}^n;\mathbb{R}^m)}$$

and the fractional Sobolev space

$$H^s(\mathbb{R}^n;\mathbb{R}^m) = \left\{ u \in L^2(\mathbb{R}^n;\mathbb{R}^m) \;\middle|\; \|u\|_{H^s(\mathbb{R}^n;\mathbb{R}^m)} < \infty \right\}$$

- It is well-known that $\mathcal{S}(\mathbb{R}^n)^n$ is dense in $H^s(\mathbb{R}^n;\mathbb{R}^n)$.

- Moreover, for each $F \in \mathcal{S}(\mathbb{R}^n)^n$,

$$\| \mathbb{P}(F) \|_{H^s(\mathbb{R}^n;\mathbb{R}^n)} \leq C \| F \|_{H^s(\mathbb{R}^n;\mathbb{R}^n)}$$

(See exercises at the end)

- Hence, $\mathbb{P}$ can be continuously extended to $H^s(\mathbb{R}^n;\mathbb{R}^n)$.

- Similarly, $\Delta^{-1}\mathrm{div}$ can be extended to a map from $H^s(\mathbb{R}^n;\mathbb{R}^n) \longrightarrow \{u \mid \nabla u \in H^s(\mathbb{R}^n;\mathbb{R}^n)\}$.

Existence — Leray Systems

In $\mathbb{R}^n \setminus \mathbb{Z}^n$: Expand F, v, p in terms of their Fourier series:

$$F(x) = \sum_{k \in \mathbb{Z}^n} \hat{F}(k)\, e^{2\pi i k \cdot x} \qquad \hat{F}(k) = \int_{\mathbb{R}^n \setminus \mathbb{Z}^n} F(x)\, e^{-2\pi i k \cdot x}\, dx$$

$$v(x) = \sum_{k \in \mathbb{Z}^n} \hat{v}(k)\, e^{2\pi i k \cdot x} \qquad \hat{v}(k) = \int_{\mathbb{R}^n \setminus \mathbb{Z}^n} v(x)\, e^{-2\pi i k \cdot x}\, dx$$

$$p(x) = \sum_{k \in \mathbb{Z}^n} \hat{p}(k)\, e^{2\pi i k \cdot x} \qquad \hat{p}(k) = \int_{\mathbb{R}^n \setminus \mathbb{Z}^n} p(x)\, e^{-2\pi i k \cdot x}\, dx$$

Since F, v, p are smooth, $\hat{F}, \hat{v}, \hat{p}$ are rapidly decreasing on $\mathbb{R}^n$.

$$\begin{cases} v = F - \nabla p \\ \operatorname{div} v = 0 \end{cases} \Longrightarrow \begin{cases} \hat{v}(k) = \hat{F}(k) - 2\pi i\, k\, \hat{p}(k) \\ 2\pi i\, k \cdot \hat{v}(k) = 0 \end{cases}$$

for each $k \in \mathbb{Z}^n$.

We have a decoupled system of vector eqns!

Taking the inner product of the first equation w/ k gives:

$$0 = k \cdot \hat{F}(k) - 2\pi i |k|^2 \hat{p}(k)$$

If $k \neq 0$ then

$$\hat{p}(k) = \frac{k \cdot \hat{F}(k)}{2\pi i |k|^2}, \qquad \hat{v}(k) = \hat{F}(k) - k\left(\frac{k \cdot \hat{F}}{|k|^2}\right)$$

If $k = 0$ then $\hat{v}(0) = \hat{F}(0)$ & $\hat{p}(0)$ is arbitrary.

Thus,

$$p(x) = C + \sum_{k \in \mathbb{Z}^n \setminus \{0\}} \frac{k \cdot \hat{F}(k)}{2\pi i \, |k|^2} e^{2\pi i k \cdot x}$$

$$v(x) = \hat{F}(0) + \sum_{k \in \mathbb{Z}^n \setminus \{0\}} \left(\hat{F}(k) - k \left(\frac{k \cdot \hat{F}(k)}{|k|^2} \right) \right) e^{2\pi i k \cdot x}$$

Again, $\quad p = C + \Delta^{-1} \operatorname{div} F$

$$v = F - \nabla \Delta^{-1} \operatorname{div} F$$

Exercises

Define

$$H_{df} = \left\{ u \in L^2(\mathbb{R}^3;\mathbb{R}^3) \;\middle|\; \overbrace{\int_{\mathbb{R}^3} u\cdot\nabla\varphi\,dx = 0}^{\operatorname{div} u = 0 \text{ in distribution sense}} \text{ for all } \varphi \in C_0^\infty(\mathbb{R}^3) \right\}$$

$$H_{cf} = \left\{ u \in L^2(\mathbb{R}^3;\mathbb{R}^3) \;\middle|\; \underbrace{\int_{\mathbb{R}^3} u\cdot \operatorname{curl}\phi\,dx = 0}_{\operatorname{curl} u = 0 \text{ in distribution sense}} \text{ for all } \phi \in (C_0^\infty(\mathbb{R}^3))^3 \right\}$$

(a) Show H_{df}, H_{cf} are closed in $L^2(\mathbb{R}^3;\mathbb{R}^3)$ and

$$L^2(\mathbb{R}^3;\mathbb{R}^3) = H_{df} \oplus H_{cf}$$

(b) Show that on $L^2(\mathbb{R}^3;\mathbb{R}^3)$, $\mathbb{P}$ is the orthogonal projection to H_{df}

(c) Show that $\mathbb{P}$ is a non-expansive map on $H^s(\mathbb{R}^3;\mathbb{R}^3)$, $\forall s \geq 0$.