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On the Uniqueness of Conformal Metrics with Prescribed Curvature

JACK THOMPSON

SUPERVISED BY
DR. ARTEM PULEMOTOV

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Abstract

Let (M, g) be a complete, connected Riemannian manifold. The purpose of this thesis is to investigate the possibility of finding metrics $\hat{g}$ with the same curvature as g, and in the same conformal class. We consider curvature of the form

$$\operatorname{Ric} g + \kappa S g + \Lambda g$$

where Ric and S denote the Ricci and Scalar curvature respectively, and κ, Λ are real constants. For $\Lambda = 0$, we prove that $\hat{g}$ is homothetic to g when $\kappa \neq -1/n$ or when $\kappa = -1/n$ and M is not conformally diffeomorphic to a space of constant curvature or a product of $\mathbf{R}$ and a complete $(n-1)$-dimensional manifold. We also give explicit examples of non-homethetic metrics when $\kappa = -1/n, \Lambda = 0$ and M is conformally diffeomorphic to one of these spaces. For $\Lambda \neq 0$, we prove $g = \hat{g}$ if M is not conformally diffeomorphic to one of these spaces.

Moreover, we consider conformal metrics with the same cross curvature on a 3 dimensional manifold with positive (negative) sectional curvature. We give the general transformation of the cross curvature under a conformal map. We also prove that if M is an Einstein space and g, $\hat{g}$ have the same cross curvature then $g = \hat{g}$. This is related to a realizability conjecture made by R. Hamilton regarding existence and uniqueness of metrics on $\mathbf{S}^3$ with prescribed cross curvature.

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Contents

Chapter 1

Introduction

Historically, geometry has been studied for many reasons. One such reason was for *cartography*, the art of drawing maps, which was driven by the need for reliable maps in order to conduct trade. In 1569, a Flemish cartographer by the name of Gerardus Mercator proposed a map of the world, known now as the Mercator projection, which would revolutionise navigation. The Mercator projection was so useful that it has become the "standard map of the world" and is likely what most people imagine when they think of a map of the world. The great property that the Mercator projection had which

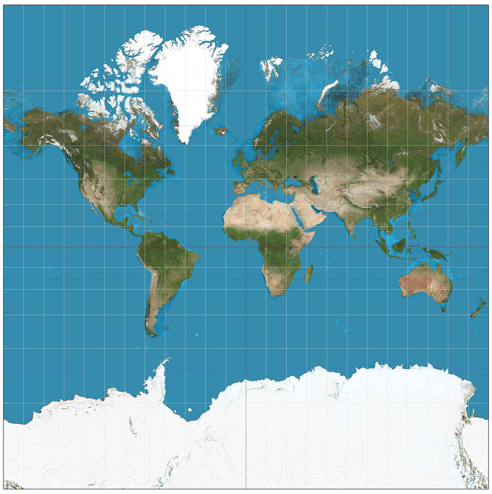

Figure 1.1: The Mercator projection

other maps didn't was that the angle between any two intersecting lines on the map was the same as the angle between the corresponding lines in real life, that is, *angles are preserved* under the transformation. A transformation with this property, such as the Mercator projection which is a transformation from the sphere to the plane, is known as a *conformal transformation*. The fact that the Mercator projection is a conformal transformation meant that a sailor wishing to travel from A to B simply had to work out the angle between a straight line from A to B and the line going North on the map, then travel at that constant bearing throughout their entire journey. Even though the sailor would not be travelling on the shortest path, this was still very attractive because of the increased reliability of navigation.

The mathematical study of conformal transformation, such as the Mercator projection, is known as *conformal geometry*. Conformal geometry has many applications in both theoretical settings and in more applied settings including in complex analysis, harmonic analysis, two-dimensional fluid flow, computer graphics, computer vision, geometric modelling, medical imaging, wireless sensor networks and general relativity, see for example [6, 15, 16]. For this reason the subject has been studied in great detail. One benefit of studying conformal maps is that often it is much easier to compute geometric quantities such as curvature than it is if one requires that area, say, is preserved. This has the downfall that conformal maps often distort other geometry quantities such as area and curvature. We can see this in Mercator projection where areas are dramatically distorted; Greenland appears to be the same size as Africa when in reality Africa is approximately 14 times bigger than Greenland. Mathematically, conformal geometry takes place on a Riemannian manifold M (a space that "locally" looks like Euclidean space) endowed with a metric g. A conformal transformation corresponds to introducing a new metric $\hat{g} = \lambda g$ where λ is a smooth, positive function from M to $\mathbf{R}$. Here we say that $\hat{g}$ is *conformal* to g. The simplest example of a conformal transformation is the rescaling of a shape, which is the case λ is constant. The angles between intersecting lines stay the same while areas are increased/decreased proportional to the scale factor. Such a transformation is known as a *homothety*.

The purpose of this thesis is to investigate the uniqueness of Riemannian metrics in a conformal class of metrics with prescribed curvature. The study of prescribing curvature is not new and has been addressed for many different notions of curvature. Possibly the most famous example is the *Yamabe problem* which asks if every Riemannian metric on a compact manifold without boundary is conformal to one of constant scalar curvature. This problem was famously introduced by H. Yamabe (from which the problem gets its name) in [24] where he claimed to have found a solution, but contained a fatal error found eight years later by N. Trudinger in [21]. Since then the answer to the Yamabe problem has been shown to be yes by the joint efforts of N. Trudinger, T. Aubin and R. Schoen. See [3] for a full exposition. However, solutions to the problem are not, in general, unique. For example, the sphere admits infinitely many metrics that are conformal to its standard metric that preserve the scalar curvature, see [2, p.588], [1, p.293]. The Yamabe problem has also been studied on compact manifolds with boundary and on noncompact, complete manifolds.

Another classical notion of curvature is the Ricci curvature. The prescribed Ricci curvature problem is a fundamental open question in Riemannian geometry. Ricci curvature has the property that it is invariant under rescaling which means that homothetic metrics have the same Ricci curvature. This has the implication that conformal metrics with the same Ricci curvature can at most be unique up to homothety. Uniqueness of conformal Riemannian metrics (up to homothety) with prescribed Ricci curvature has been proven for compact manifolds without boundary in [22, 23] and on noncompact, complete manifolds in [17]. In Chapter 3, we consider the uniqueness of conformal metrics with prescribed curvature of the form:

$$\operatorname{Ein} g := \operatorname{Ric} g + \kappa S g + \Lambda g \tag{1.1}$$

where Ric and S denote the Ricci and scalar curvature of g respectively and κ, Λ are real constants. We refer to tensors of the form of (1.1) as *trace-adjusted Ricci tensors*. Some common examples of trace-adjusted Ricci tensors are the Ricci curvature, Einstein tensor with cosmological constant and the Schouten tensor. Trace-adjusted Ricci tensors were studied by E. Delay in [9, 10, 11, 12] where he thoroughly studied their local and global invertibility in both compact and noncompact settings. The uniqueness of conformal metrics with a prescribed trace-adjusted Ricci tensor depends on the values of κ and Λ.

Let (M, g) be a complete, connected Riemannian manifold with dimension $n \geqslant 3$ and $C_+^\infty(M)$ be the set of all smooth, positive functions from M to $\mathbf{R}$. Moreover, we denote by $\mathbf{S}^n, \mathbf{R}^n, \mathbf{H}^n$ the sphere, Euclidean space and hyperbolic space with their standard metrics respectively. In this thesis, we prove the following results.

Theorem 1.1. *If $\hat{g} = \varphi^{-2} g$, $\varphi \in C_+^\infty(M)$, and $\operatorname{Ein} g = \operatorname{Ein} \hat{g}$, $\kappa \neq -\frac{1}{n}, \Lambda = 0$, then φ is a constant.*

Theorem 1.2. *Suppose $\hat{g} = \varphi^{-2} g$, $\varphi \in C_+^\infty(M)$, and $\operatorname{Ein} g = \operatorname{Ein} \hat{g}$, $\kappa = -\frac{1}{n}, \Lambda = 0$. If M is not conformally diffeomorphic to $\mathbf{S}^n$, $\mathbf{R}^n$, $\mathbf{H}^n$ or $\mathbf{R} \times M_*$, where M_* is a complete $(n-1)$-dimensional manifold, then φ is a constant.*

Theorem 1.3. *Suppose $\hat{g} = \varphi^{-2} g$, $\varphi \in C_+^\infty(M)$, and $\operatorname{Ein} g = \operatorname{Ein} \hat{g}$, $\kappa \neq -\frac{1}{n}, \Lambda \neq 0$. Moreover, suppose*

(i) M is not conformally diffeomorphic to $\mathbf{H}^n$ or $\mathbf{R} \times M_*$, where M_* is a complete $(n-1)$-dimensional manifold, and $\frac{\Lambda}{\kappa n+1} > 0$; or

(ii) M is not conformally diffeomorphic to $\mathbf{S}^n$, $\mathbf{H}^n$ or $\mathbf{R} \times M_*$, where M_* is a complete $(n-1)$-dimensional manifold, and $\frac{\Lambda}{\kappa n+1} < 0$.

Then $\varphi = 1$.

In Chapter 4, we consider the uniqueness of conformal metrics with prescribed cross curvature. Cross curvature is a type of curvature introduced by R. Hamilton and B. Chow in [8] to produce a geometric flow proof that all manifolds of dimension 3 with nonnegative sectional curvature admit a hyperbolic metric, that is, a metric with constant sectional curvature equal to -1. We derive the transformation of the cross curvature, the trace of the cross curvature and the traceless cross curvature under a conformal change of metric. Moreover, we prove the following:

Theorem 1.4. *Suppose (M^3, g) is an Einstein manifold with positive sectional curvature and $\hat{g} = \frac{1}{\varphi^2} g$, $\varphi \in C_+^\infty(M)$. If $X_{ij} = \hat{X}_{ij}$ then $\varphi = 1$.*

Here X_{ij} denotes the components of the cross curvature. This is related to a realizability conjecture made by R. Hamilton regarding existence and uniqueness of metrics on $\mathbf{S}^3$ with prescribed cross curvature.

Chapter 2

Preliminaries

2.1 Riemannian Geometry

2.1.1 Smooth Manifolds

We begin with some introductory theory on Riemannian manifolds. Intuitively, a manifold is a space that locally looks like Euclidean space $\mathbf{R}^n$. For example, the Earth is a 2 dimensional manifold since at at every point it appears to be a flat plane ($\mathbf{R}^2$). This example demonstrates a common theme in the theory of manifolds which is that, even though they look like $\mathbf{R}^n$ up close, they can look wildly different from $\mathbf{R}^n$ far away.

Definition. An n dimensional *smooth manifold* is a second countable, Hausdorff topological space M and a collection $\mathscr{F} = \{(\psi_\alpha, \Omega_\alpha)\}$ where each $\Omega_\alpha \subset M$ is open and ψ_α is an injective map from Ω_α to $\mathbf{R}^n$ such that

(i) $\bigcup\limits_\alpha \Omega_\alpha = M$;

(ii) If $\Omega_\alpha \cap \Omega_\beta \neq \emptyset$ then the map $\psi_\alpha \circ \psi_\beta^{-1} : \psi_\beta(\Omega_\alpha \cap \Omega_\beta) \subset \mathbf{R}^n \to \psi_\alpha(\Omega_\alpha \cap \Omega_\beta) \subset \mathbf{R}^n$ is smooth; and

(iii) $\mathscr{F} = \{(\psi_\alpha, \Omega_\alpha)\}$ is maximal in the sense that if $\Omega \subset M$ is open, $\psi : \Omega \to \mathbf{R}^n$ is injective and $\psi \circ \psi_\alpha^{-1} : \psi_\alpha(\Omega \cap \Omega_\alpha) \to \psi(\Omega \cap \Omega_\alpha)$, $\psi_\alpha \circ \psi^{-1} : \psi(\Omega \cap \Omega_\alpha) \to \psi_\alpha(\Omega \cap \Omega_\alpha)$ are smooth for all $(\psi_\alpha, \Omega_\alpha) \in \mathscr{F}$ then $(\psi, \Omega) \in \mathcal{F}$.

The pair $(\psi, \Omega) \in \mathscr{F}$ with $p \in \Omega$ are referred to as a *chart* at p and $\mathscr{F}$ is a *smooth structure* on M.

Remarks. 1. Assumption (iii) is purely technical since any collection $\mathscr{F} = \{(\psi_\alpha, \Omega_\alpha)\}$ satisfying (i) and (ii) can always be extended to a maximal one $\tilde{\mathscr{F}}$. Indeed, $\tilde{\mathscr{F}}$ can be constructed by taking the union of all collections $\{(\psi, \Omega)\}$ which contain $\mathscr{F}$ and satisfy (i) and (ii).

2. A classical result in topology says that if U is an open subset of $\mathbf{R}^n$, V is an open subset of $\mathbf{R}^m$ and $f : U \to V$ is a homeomorphism then $n = m$. This implies that a manifold cannot be both n dimensional and m dimensional for $n \neq m$.

Definition. Suppose (ψ, Ω) is a chart of a smooth manifold M with dimension n. Since ψ maps into $\mathbf{R}^n$ we can write $\psi = (x^1, \ldots, x^n)$ where $x^i : \Omega \to \mathbf{R}$. Then $\{x^i\}_{i=1}^n$ are called *local coordinates*.

Example 2.1. The simplest example of an n dimensional manifold is the Euclidean space $\mathbf{R}^n$ with smooth structure $\mathscr{F} = \{(\mathbf{R}^n, \mathrm{id})\}$.

Example 2.2. Let $\mathbf{S}^n$ be the set of points $p = (p^1, \ldots, p^{n+1}) \in \mathbf{R}^{n+1}$ such that $\sum_{i=1}^{n+1}(p^i)^2 = 1$ and let the topology be induced from $\mathbf{R}^{n+1}$. Then $\mathbf{S}^n$ can be given a smooth structure. An example of such a smooth structure is defined via the *stereographic projections*; let $N = (0, \ldots, 0, 1) \in \mathbf{R}^{n+1}$ and define $\psi_N : \mathbf{S}^n - \{N\} \to \mathbf{R}^n$ by

$$p \mapsto \left(\frac{p^1}{1 - p^{n+1}}, \ldots, \frac{p^n}{1 - p^{n+1}}\right).$$

The map ψ_N is called the *stereographic projection from the north pole* and takes the point $p \in \mathbf{S}^n - \{N\}$ to the point at which the straight line from N to p intersects the hyperplane $\{p^{n+1} = 0\}$. Analogously, $S = (0, \ldots, -1) \in \mathbf{R}^{n+1}$ and the *stereographic projection from the south pole* is $\psi_S : \mathbf{S}^n - \{S\} \to \mathbf{R}^n$ defined by

$$p \mapsto \left(\frac{p^1}{1 + p^{n+1}}, \ldots, \frac{p^n}{1 + p^{n+1}}\right).$$

The pairs $(\psi_N, \mathbf{S}^n - \{N\})$ and $\{\psi_S, \mathbf{S}^n - \{S\})$ are compatible charts, so can be extended to a smooth structure on $\mathbf{S}^n$.

Example 2.3. Let M be a smooth manifold and U an open subset of M. If $\mathscr{F}$ is the smooth structure on M then U is a smooth manifold with smooth structure given by $\mathscr{F}' = \{(\psi|_{U\cap\Omega}, U \cap \Omega) \,|\, (\psi, \Omega) \in \mathscr{F}\}$.

Example 2.4. Let M and M' be smooth manifolds with dimension m and n respectively. Let $\mathscr{F}$ and $\mathscr{F}'$ be the smooth structure on M and M' respectively. Then $M \times M'$ is a smooth manifold with smooth structure given by $\{(\psi \times \psi', \Omega \times \Omega') \,|\, (\psi, U) \in \mathscr{F}, (\psi', \Omega') \in \mathscr{F}\}$ where $\psi \times \psi' : \Omega \times \Omega' \to \mathbf{R}^m \times \mathbf{R}^n$ is defined by $(p, q) \mapsto (\psi(p), \psi'(q))$. It can be shown $M \times M'$ has dimension $m + n$.

Definition. Let M and M' be smooth manifolds. A continuous function $f : M \to M'$ is *differentiable* at $p \in M$ if there exist charts (ψ, Ω) and (ψ', Ω') of p and $f(p)$ respectively such that

$$\psi' \circ f \circ \psi^{-1} : \psi(\Omega) \to \psi'(\Omega')$$

is differentiable at $\psi(p)$. If f is differentiable at every point in M we simply say f is differentiable. The set of differentiable function from M to $\mathbf{R}$ is denoted $C^\infty(M)$ and

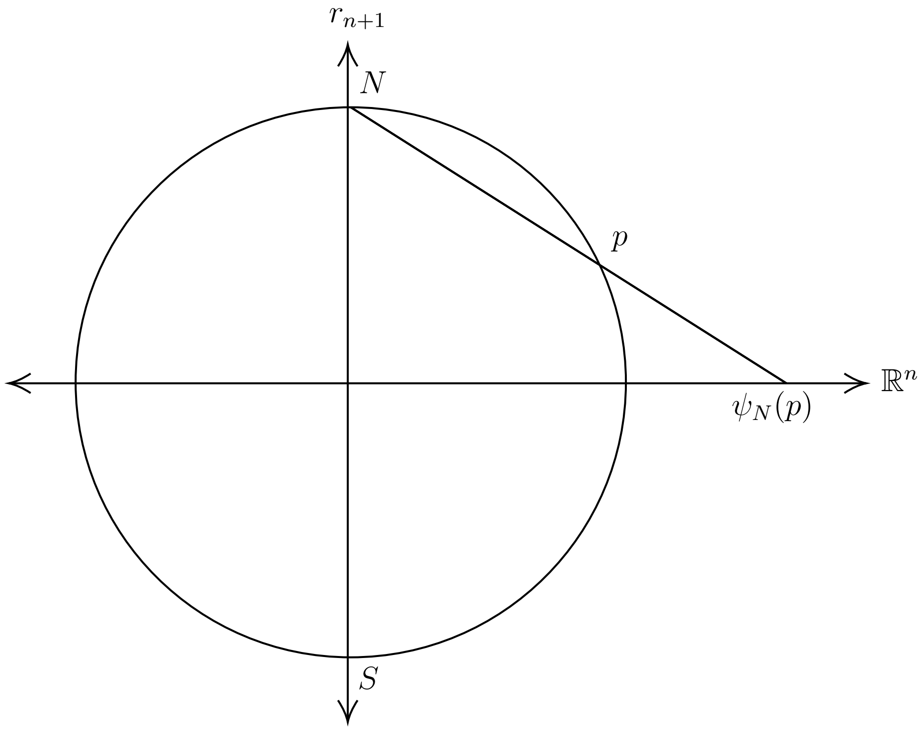

Figure 2.1: The stereographic projection.

the set of functions from M to $\mathbf{R}$ which are differentiable in a neighbourhood of a point $p \in M$ is denoted $C^\infty(M, p)$.

A differentiable function f is a *diffeomorphism* if f is also a bijection and $f^{-1} : M' \to M$ is differentiable.

Remark. The definition of differentiability does not depend on the choice of charts.

2.1.2 Tangent Vectors and Vector Fields

Before we discuss the notion of a tangent space to a point it will be useful to motivate the definition. Suppose $M = \mathbf{R}^n$ and we have a smooth curve $\alpha : (-\varepsilon, \varepsilon) \to \mathbf{R}^n$ such that $\alpha(0) = p \in M$. We may write

$$\alpha(t) = (x^1(t), \ldots, x^n(t))$$

for functions $x^i : (-\varepsilon, \varepsilon) \to \mathbf{R}$. The tangent of α at p will be given by

$$\alpha'(0) = ((x^1)'(0), \ldots, (x^n)'(0)) =: v.$$

If $f \in C^\infty(M, p)$ then directional derivative of f in direction v is given by

$$\begin{aligned} \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 (f \circ \alpha)(t) &= \sum_{i=1}^n \left.\frac{\partial f}{\partial x^i}\right|_p \left.\frac{\mathrm{d}x^i}{\mathrm{d}t}\right|_0 \\ &= \sum_{i=1}^n v_i \left.\frac{\partial f}{\partial x^i}\right|_p \end{aligned}$$

We can think of the tangent vector v as being a differential operator mapping f to

$$v(f) = \sum_{i=1}^{n} v_i \left.\frac{\partial f}{\partial x^i}\right|_p \in \mathbf{R}.$$

This operator is a linear map form $C^\infty(M)$ to $\mathbf{R}$ and is a derivation, that is, for all $f, g \in C^\infty(M)$, $v(fg) = v(f)g + fv(g)$.

Definition. A *tangent vector* to M at $p \in M$ is a function $v : C^\infty(M, p) \to \mathbf{R}$ that satisfies for all $a, b \in \mathbf{R}$, $f, g \in C^\infty(M, p)$:

(i) $v(af + bg) = av(f) + bv(f)$; and

(ii) $v(fg)(p) = v(f)(p)g(p) + f(p)v(g)(p)$.

The *tangent space* to M at p is the set of all tangent vectors, denoted T_pM.

Remark. If we defined $(v + w)(f) = v(f) + w(f)$ and $(av)(f) = a \cdot v(f)$ then condition (i) in the above definition implies T_pM is a vector space over $\mathbf{R}$.

Definition. If $(\Omega, \psi = (x^1, \ldots, x^n))$ is a chart containing p then we define the tangent vector

$$\left.\frac{\partial}{\partial x^i}\right|_p f = \left.\frac{\partial}{\partial r^i}\right|_{\varphi(p)} (f \circ \psi^{-1}) \tag{2.1}$$

for all $f \in C^\infty(M, p)$. Here r^j are simply the standard coordinates in $\mathbf{R}^n$.

We interpret (2.1) as the directional derivative of f at p in the x_i coordinate. We will use the notation

$$\left.\frac{\partial f}{\partial x_i}\right|_p = \left.\frac{\partial}{\partial x_i}\right|_p (f).$$

Remark. The set

$$\left\{ \left.\frac{\partial}{\partial x^i}\right|_p \right\}$$

forms a basis for T_pM and if $v \in T_pM$ then

$$v = \sum_{i=1}^{n} v(x_i) \left.\frac{\partial}{\partial x_i}\right|_p .$$

Definition. Let $f : M \to M'$ be a differentiable map and $p \in M$. The *differential map* of f at p is a function $\mathrm{d}f_p : T_pM \to T_{f(p)}M'$ defined by

$$\mathrm{d}f_p(v)(g) = v(f \circ g),$$

for $v \in T_pM$ and $g \in C^\infty(M', f(p))$.

Remark. The differential $\mathrm{d}f_p$ is linear for each $p \in M$.

Observe that if $M' = \mathbf{R}$ then for $f \in C^\infty(M)$, $\mathrm{d}f_p : T_pM \to T_{f(p)}\mathbf{R} \cong \mathbf{R}$. This leads to the following definition.

Definition. Let $\gamma : I \to M$, I is an open interval in $\mathbf{R}$ be a differentiable curve on M. We define the tangent vector to γ at $t \in I$ by

$$\gamma'(t) = \mathrm{d}\gamma_t \left(\left.\frac{\mathrm{d}}{\mathrm{d}r}\right|_t \right).$$

We have discussed in detail the notion of a tangent vector at a fixed point $p \in M$. However, what happens when we allow p to vary? This leads to the notion of a *vector field*. To define a vector field, we first need the notion of the tangent bundle.

Definition. Let M be a smooth manifold. The *tangent bundle* of M is the set

$$TM = \{(p, v) \,:\, p \in M, v \in T_pM\}.$$

The projection $\pi : TM \to M$ is defined by $(p, v) \mapsto p$.

There is a natural smooth structure on TM inherited from M. With this smooth structure TM is a smooth manifold with $\dim(TM) = 2 \cdot \dim M$.

Definition. A vector field X in M is a map $M \to TM$ such that

$$\pi \circ X = \mathrm{id}_M,$$

that is, $X_p \in T_pM$ for all $p \in M$.

A vector field is *differentiable* if the map $X : M \to TM$ is differentiable. The set of differentiable vector fields on a smooth manifold M is denoted $\mathfrak{X}(M)$.

If $f : M \to \mathbf{R}$ is a differentiable function then $X(f)$ is the function from $M \to \mathbf{R}$ given by $p \mapsto X_p(f)$.

Lemma 2.1. *Let X be a vector field on M. Then the following statements are equivalent.*

(i) $X \in \mathfrak{X}(M)$.

(ii) *If $(\Omega, \varphi = (x^1, \ldots, x^n))$ is a chart of M and $\{a_i\}$ are the collection of functions such that for all $p \in \Omega$,*

$$X(p) = \sum_{i=1}^{n} a_i(p) \left.\frac{\partial}{\partial x_i}\right|_p$$

then $a_i : M \to \mathbf{R}$ are differentiable.

(iii) *If $\Omega \subset M$ is open and $f : \Omega \to \mathbf{R}$ is differentiable then $X(f) : \Omega \to \mathbf{R}$ is differentiable.*

2.1.3 Riemannian Manifolds

So far we have only discussed the *differentiable* structure of smooth manifolds, but now we would like to discuss the *geometry* of smooth manifolds. In order to do this we need some notion of "distance". This is achieved by introducing a *Riemannian metric*. Through the Riemannian metric we can define geometric quantities such as curvatures, lengths, areas and angles.

Definition. A *Riemannian metric* on a smooth manifold M is a correspondence which associates to each point $p \in M$ an inner product $g_p = \langle \cdot, \cdot \rangle_p$ on the tangent space T_pM such that for all $X, Y \in \mathfrak{X}(M)$ the map $p \mapsto \langle X_p, Y_p \rangle_p$ is smooth. Then (M, g) is called a *Riemannian manifold*.

If $\{x^i\}$ are local coordinates then the components g_{ij} of the metric g are given by

$$(g_{ij})_p = g_p\left(\left.\frac{\partial}{\partial x^i}\right|_p, \left.\frac{\partial}{\partial x^j}\right|_p \right).$$

Example 2.5. The simplest example of a Riemannian manifold is $\mathbf{R}^n$ with metric given by the standard dot product. Explicitly, at each point $T_p\mathbf{R}^n \cong \mathbf{R}^n$, so define

$$g_p(u, v) = u \cdot v.$$

In the standard local coordinates, $g_{ij} = \delta_{ij}$ where δ_{ij} is the Kronecker delta. Metrics of this form are called *Euclidean* and will be denoted by g_0.

Example 2.6. Let $M = \mathbf{S}^n$. Since $\mathbf{S}^n \subset \mathbf{R}^{n+1}$ for each $p \in \mathbf{S}^n$, $T_p\mathbf{S}^n \subset T_p\mathbf{R}^{n+1} \cong \mathbf{R}^{n+1}$. The standard metric on $\mathbf{S}^n$, denoted g_1, is then defined by using the metric inherited from $\mathbf{R}^{n+1}$.

Example 2.7. Let $M = \mathbf{R}^n_+ = \{(r^1, \dots, r^n) \in \mathbf{R}^n \,|\, r^n > 0\}$. Since $\mathbf{R}^n_+$ is an open subset it can be given a smooth structure inherited from $\mathbf{R}^n$, see Example 2.3. In the standard coordinates of $\mathbf{R}^n_+$ a metric g_{-1} is defined by

$$(g_{-1})_{ij} = \frac{\delta_{ij}}{(r^n)^2}.$$

This is known as the *hyperbolic metric*. Then $(\mathbf{R}^n_+, g_{-1})$ is called *Hyperbolic Space*, denoted by $\mathbf{H}^n$.

Example 2.8. Let (M, g) and (M', g') be Riemannian manifolds with dimension m and n Recall, as in Example 2.4, $M \times M'$ can be given a smooth structure which makes in an $n+m$ dimensional manifold. One can show that at $(p, p') \in M \times M'$ the tangent space is $T_{(p,p')}(M \times M') \cong T_pM \oplus T_{p'}M'$ where $\oplus$ denotes the direct sum of vector spaces. Then $M \times M'$ is also a Riemannian manifold with metric given by

$$\tilde{g}((u, u'), (v, v')) = g(u, v) + g'(u', v')$$

for each $u, v \in T_pM$ and $u', v' \in T_{p'}M'$.

2.1.4 Connections and Curvature

Suppose (M, g) is a smooth Riemannian manifold of dimension n. Let $\mathfrak{X}(M)$ denote the set of all smooth vector fields on M, $C^\infty(M)$ the set of smooth functions from M to $\mathbf{R}$ and $C^\infty_+(M)$ the set of smooth, positive functions from M to $\mathbf{R}$.

Definition. An *affine connection* on M is a map $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ denoted by $(X, Y) \mapsto \nabla_X Y$ such that for all $X, Y, Z \in \mathfrak{X}(M)$, $f, g \in C^\infty(M)$

1. $\nabla_X(Y + Z) = \nabla_X Y + \nabla_X Z$
2. $\nabla_{fX+gY} = f\nabla_X Z + g\nabla_Y Z$
3. $\nabla_X(fY) = f\nabla_X Y + X(f)Y$.

Let $[\cdot, \cdot]$ denote the Lie Bracket on $\mathfrak{X}(M)$ defined by $[X, Y]_p = X_p(Y) - Y_p(X)$.

Theorem 2.2 (Fundamental Theorem of Riemannian Geometry)**.** *Suppose (M, g) is a smooth Riemannian manifold. There exists a unique affine connection ∇ satisfying*

1. $[X, Y] = \nabla_X Y - \nabla_Y X$ *and*
2. $\nabla_X g(Y, Z) = g(\nabla_X Y, Z) + g(X, \nabla_X Z)$

for all $X, Y, Z \in \mathfrak{X}(M)$.

Definition. The connection defined in Theorem 2.2 is called the *Levi-Cevita Connection* on (M, g).

From now on ∇ will always denote the Levi-Cevita Connection on (M, g). There are four main examples of curvature: Riemannian curvature, Ricci curvature, Scalar curvature and Sectional curvature. For completeness, we have given their definitions here.

Definition. The *(1,3)-Riemannian curvature* $R : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ is defined by

$$R(X, Y)Z = \nabla_{[X,Y]}Z - \nabla_X\nabla_Y Z + \nabla_Y\nabla_X Z.$$

The *(0,4)-Riemannian curvature* $\mathrm{Rm} : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathbf{R}$ is then defined by

$$\mathrm{Rm}(X, Y, Z, W) = g(R(X, Y)Z, W).$$

In local coordinates, the components of (1,3)-Riemannian curvature and the (0,4)-Riemannian curvature are denoted ${R_{ijk}}^l$ and R_{ijkl} respectively.

Definition. The *Ricci curvature* $\mathrm{Ric} : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathbf{R}$ is defined by

$$\mathrm{Ric}(X,Y) = \mathrm{tr}\{Z \mapsto R(X,Z)Y\}.$$

In local coordinates, the components of the Ricci curvature are denoted R_{ij} and are equal to ${R_{ikj}}^k = g^{kl}R_{ikjl}$ where g^{ij} are the components of the inverse of the metric g and the summation convention is employed.

Definition. The *scalar curvature* $S : M \to \mathbf{R}$ is given by

$$S = \mathrm{tr}_g \mathrm{Ric}$$

where tr_g means the trace with respect to the metric g. In local coordinates, S is equal to $g^{ij}R_{ij}$.

Definition. Fix a point $p \in M$ and let Π be a 2 dimensional subspace of T_pM. If $v, w \in T_pM$ span Π then the *sectional curvature* of Π, $K(p)$, is given by

$$K(p) = \frac{\mathrm{Rm}(u,v,u,v)}{g(u,u)g(v,v) - g(u,v)^2}.$$

It is well-known that the only Riemannian manifolds (up to isometry and rescaling) with constant sectional curvature are $(\mathbf{S}^n, g_1)$, $(\mathbf{R}^n, g_0)$ and $(\mathbf{H}^n, g_{-1})$ which have sectional curvature 1,0 and -1 respectively.

2.2 Conformal Geometry

Conformal geometry is the study of *conformal maps* which are functions between manifolds that *preserve angles*. The theory of conformal maps has found great success in many areas and as such is usually a standard topic in most undergraduate mathematics curriculum. They have applications in pure and applied mathematics and physics including complex analysis, harmonic analysis, two-dimensional fluid flow, computer graphics, computer vision, geometric modelling, medical imaging, wireless sensor networks and general relativity, see for example [6, 15, 16]. Here we will focus on conformal maps in the context of Riemannian geometry.

In $\mathbf{R}^n$, the angle between two vectors has a clear geometric meaning. If $u, v \in \mathbf{R}^n$ then $u \cdot v$ is the length of the projection of u onto v, and basic trigonmetry implies that the angle θ between u and v satisfies the well know formula

$$u \cdot v = |u||v| \cos\theta.$$

Let (M, g) denote a smooth Riemannian manifold. Motivated by the case in $\mathbf{R}^n$ we have the following definition:

Definition. The *angle* between two nonzero vector fields $X, Y \in \mathfrak{X}(M)$ is defined to be

$$\theta(X,Y) = \arccos\left(\frac{g(X,Y)}{\sqrt{g(X,X)}\sqrt{g(Y,Y)}}\right).$$

Note that the angle depends on the metric. Then we can define conformal metrics.

Definition. Let $g, \hat{g}$ be metrics on M and $\theta, \hat{\theta}$ be the corresponding angle functions. Then g and $\hat{g}$ are said to be *conformal* if $\theta(X,Y) = \hat{\theta}(X,Y)$ for all nonzero $X, Y \in \mathfrak{X}(M)$. Moreover, two Riemannain manifolds (M, g) and (M', g') are *conformally diffeomorphic*, sometimes referred to as *conformally equivalent*, if there exists a diffeomorphism $f : M \to M'$ such that f^*g' is conformal to g.

This provides an intuitive understanding of conformal metrics, but in practice this definition can be difficult to work with. Observe that if $\hat{g} = \lambda g$ for some positive function $\lambda : M \to \mathbf{R}$ then by a simple calculation $\hat{\theta} = \theta$. If $\hat{\theta} = \theta$ then under the change of metric $g \to \hat{g}$ vectors are "rescaled" so we might expect the reverse implication to hold. This is indeed true and is the subject of the following theorem.

Theorem 2.3. *Two metrics g and $\hat{g}$ on M are conformal if and only if $\hat{g} = \lambda g$ for some smooth positive function $\lambda : M \to \mathbf{R}$.*

Proof. If $\hat{g} = \lambda g$ for some smooth positive function $\lambda : M \to \mathbf{R}$ then for each nonzero $X, Y \in \mathfrak{X}(M)$,

$$\begin{aligned}\hat{\theta}(X,Y) &= \arccos\left(\frac{\lambda g(X,Y)}{\sqrt{\lambda g(X,X)}\sqrt{\lambda g(Y,Y)}}\right)\\ &= \theta(X,Y).\end{aligned}$$

Now suppose that g and $\hat{g}$ are conformal on M and fix $p \in M$. About p we may choose coordinates $\{x^i\}_{i=1}^n$ such that $\{e_i\}_{i=1}^n$ with $e_i = \frac{\partial}{\partial x^i}|_p$ form an orthonormal basis for T_pM with respect to g. Since $\hat{\theta} = \theta$, it follows

$$\hat{g}_{ij}(p) = \delta_{ij}\sqrt{\hat{g}_{ii}(p)}\sqrt{\hat{g}_{ii}(p)}.$$

If $i \neq j$ then $\hat{g}_{ij} = 0$ so $\{e_i\}_{i=1}^n$ is orthogonal with respect to $\hat{g}$. This means $\hat{g}(p)$ is entirely determined by the values of the diagonal terms $\hat{g}_{ii}(p)$. Observe that for $i \neq j$,

$$\begin{aligned}g(e_i + e_j, e_i - e_j) &= g(e_i, e_i) - g(e_j, e_j)\\ &= 0\end{aligned}$$

so $e_i + e_j$ is orthogonal to $e_i - e_j$ with respect to g which implies that $e_i + e_j$ must also be orthogonal to $e_i - e_j$ with respect to $\hat{g}$. But then

$$\begin{aligned}0 &= \hat{g}(e_i + e_j, e_i - e_j)\\ &= \hat{g}(e_i, e_i) - \hat{g}(e_j, e_j)\end{aligned}$$

so $\hat{g}_{ii}(p) = \hat{g}_{jj}(p) = \lambda_p$ for some positive constant λ_p. Hence,

$$\hat{g}_{ij}(p) = \lambda_p \delta_{ij} = \lambda_p g_{ij}(p).$$

The point p was arbitrary, so we can define $\lambda : M \to \mathbf{R}$ by $p \mapsto \lambda_p$. All that is left to be shown is that λ is smooth. Indeed, in any coordinate patch $\lambda = \hat{g}_{ii}/g_{ii}$ which is smooth, so the theorem is proven. □

Remark. It often makes calculations easier to consider $\varphi = \frac{1}{\sqrt{\lambda}}$, so that $\hat{g} = \frac{1}{\varphi^2} g$. It is also common to let $\rho = \frac{1}{2}\log\lambda$ which gives $\hat{g} = e^{2\rho} g$.

The following are some examples of conformal changes of metric.

Example 2.9. The simplest example of a conformal change of metric is when $\hat{g} = cg$ for some constant $c > 0$. In this case, we say $\hat{g}$ is *homothetic* to g.

Example 2.10. The punctured sphere $\mathbf{S}^n - \{N\}$ is conformally diffeomorphic to $\mathbf{R}^n$ via stereographic projection. Recall that the stereographic projection $\phi_N : \mathbf{S}^n - \{N\} \subset \mathbf{R}^{n+1} \to \mathbf{R}^n$ is given by

$$p \mapsto \left(\frac{p_1}{1 - p_{n+1}}, \ldots, \frac{p_n}{1 - p_{n+1}} \right).$$

This can be shown to be a diffeomorphism. Let $f = \phi_N^{-1}$. Then for $r = \phi_N(p) \in \mathbf{R}^n$,

$$f(r) = \left(\frac{2r}{|r|^2 + 1}, \frac{|r|^2 - 1}{|r|^2 + 1} \right).$$

If g_1 is the standard metric on $\mathbf{S}^n - \{N\}$ then the pullback metric $g_* = f^* g_1$ on $\mathbf{R}^n$ is given by

$$(g_*)_r(v, w) = (g_1)_r(\mathrm{d}f_r v, \mathrm{d}f_r w)$$

for each $v, w \in T_r\mathbf{R}^n \cong \mathbf{R}^n$. Let $\{e_i\}_{i=1}^n$ denote the standard basis elements of $\mathbf{R}^n$. Then[1]

$$\begin{aligned}
(\mathrm{d}f e_i)^\alpha &= \frac{\partial f^\alpha}{\partial r^i} \\
&= \begin{cases} \frac{2\delta_i^\alpha}{|r|^2+1} - \frac{4r^\alpha r_i}{(|r|^2+1)^2}, & 1 \leqslant \alpha \leqslant n \\ \frac{4r_i}{(|r|^2+1)^2}, & \alpha = n+1 \end{cases}
\end{aligned}$$

where $r_i = \delta_{ik} r^k$. Since g_1 is the induced metric from $\mathbf{R}^{n+1}$, it follows that

$$\begin{aligned}
(g_*)_{ij} &= \delta_{\alpha\beta}(\mathrm{d}f e_i)^\alpha (\mathrm{d}f e_j)^\beta \\
&= \delta_{kl}\left(\frac{2\delta_i^k}{|r|^2+1} - \frac{4r^k r_i}{(|r|^2+1)^2} \right)\left(\frac{2\delta_j^l}{|r|^2+1} - \frac{4r^l r_j}{(|r|^2+1)^2} \right) + \frac{16 r_i r_j}{(|r|^2+1)^4} \\
&= \frac{4\delta_{ij}}{|r|^2+1}.
\end{aligned}$$

[1]Throughout this example latin indices i, j, k, l will run from 1 to n whereas greek letters α, β will run from 1 to $n+1$

This shows that $g_* = \lambda g_0$ where g_0 is the standard metric on $\mathbf{R}^n$ and

$$\lambda(r) = \frac{4}{|r|^2 + 1},$$

so $\mathbf{S}^n - \{N\}$ is conformally diffeomorphic to $\mathbf{R}^n$.

One can show that the Ricci curvature remains unchanged in Example 2.9, however, in Example 2.10 the Ricci curvature changes. This shows that Ricci curvature is invariant under homotheties. A natural question is does the converse statement hold, that is, if two conformal metrics have the same Ricci curvature, are they necessarily hometHetic? This is the subject of Chapter 3 where we address this problem for a class of curvatures of which the Ricci curvature is a part.

Chapter 3

Trace-Adjusted Ricci Tensors in a Conformal Class of Metrics

3.1 Introduction

Let (M, g) be a Riemannian manifold of dimension $n \geqslant 3$.

Definition. A *trace-adjusted Ricci tensor*[1] is a symmetric $(0,2)$-tensor of the form

$$\operatorname{Ein} g = \operatorname{Ric} g + \kappa S g + \Lambda g \tag{3.1}$$

where $\operatorname{Ric} g$, S are the Ricci and Scalar curvatures of g respectively and κ, Λ are real constants.

Tensors of this form appear in many areas of mathematics and physics. Some common examples of trace-adjusted Ricci tensors are: the Ricci curvature tensor ($\kappa = \Lambda = 0$), the traceless Ricci curvature ($\kappa = -\frac{1}{n}$, $\Lambda = 0$), the Einstein tensor with cosmological constant ($\kappa = -\frac{1}{2}$, $\Lambda \in \mathbf{R}$) and the Schouten tensor multiplied by a factor of $n-2$ ($\kappa = -\frac{1}{2(n-1)}$, $\Lambda = 0$). Trace-adjusted Ricci tensors were studied by Delay in [9, 10, 11, 12] where he thoroughly investigated their local and global invertibility in both compact and noncompact settings.

The purpose of this chapter is to investigate the uniqueness of metrics in a conformal class with prescribed trace-adjusted Ricci tensors. It is well known that conformal metrics with prescribed Ricci curvature are unique up to homothety, as was proven when M is compact, orientated and connected without boundary in [22, 23] and then when M is connected and complete in [17]. Due to the similarities between trace-adjusted Ricci tensors and the Ricci curvature, one might expect that the results of [22] and [17] generalise to (3.1).

[1] This name comes from the fact that the corresponding (1,1) tensor of $\operatorname{Ein} g$ is equal to the (1,1) Ricci curvature plus a multiple of the identity.

There are, however, some key nuances. Firstly, when $\kappa = -\frac{1}{n}, \Lambda = 0$, $\mathrm{Ein} = \mathrm{Ric}^{\circ}$, the traceless Ricci curvature. We show that the question of uniqueness in a conformal class becomes equivalent to the classification of manifolds admitting nonconstant concircular functions, which are functions whose Hessian is proportional to the metric. This second problem is, in general, a nontrivial problem, see [19, 18]. The second difference is when $\Lambda \neq 0$. In this case, the techniques used in [22] and [17] no longer apply, so alternate methods must be used.

Throughout this chapter let (M, g) be a complete, connected Riemannian manifold with dimension $n \geqslant 3$. We will denote by $C^{\infty}_{+}(M)$ the set of all smooth, positive functions from M to $\mathbf{R}$. The first theorem pertains to when $\kappa \neq -\frac{1}{n}$, $\Lambda = 0$ and recovers the result given in [17].

Theorem 3.1. *If $\hat{g} = \varphi^{-2} g$, $\varphi \in C^{\infty}_{+}(M)$, and $\mathrm{Ein}\, g = \mathrm{Ein}\, \hat{g}$, $\kappa \neq -\frac{1}{n}, \Lambda = 0$ then φ is a constant.*

Next, we consider the case $\kappa = -\frac{1}{n}$, $\Lambda = 0$. Theorem 3.2 is an application of a theorem by Tashiro, see [20, Thm 1]. For the convenience of the reader we have restated the result by Tashiro here as Theorem 3.13.

Theorem 3.2. *Suppose $\hat{g} = \varphi^{-2} g$, $\varphi \in C^{\infty}_{+}(M)$, and $\mathrm{Ein}\, g = \mathrm{Ein}\, \hat{g}$, $\kappa = -\frac{1}{n}, \Lambda = 0$. If M is not conformally diffeomorphic to $\mathbf{S}^n$, $\mathbf{R}^n$, $\mathbf{H}^n$ or $\mathbf{R} \times M_*$, where M_* is a complete $(n-1)$-dimensional manifold, then φ is a constant.*

Furthermore, we give explicit examples on $\mathbf{S}^n$, $\mathbf{R}^n$, $\mathbf{H}^n$ or $I \times M_*$ where uniqueness fails.

Now we turn to the case $\Lambda \neq 0$. We show if $\kappa = -\frac{1}{n}$ then trivially $\varphi = 1$, see the discussion before Theorem 3.8. Hence, we consider the case $\kappa \neq -\frac{1}{n}$ and $\Lambda \neq 0$. We obtain partial results and prove uniqueness in all cases except on $\mathbf{H}^n$ and $I \times M_*$.

Theorem 3.3. *Suppose $\hat{g} = \varphi^{-2} g$, $\varphi \in C^{\infty}_{+}(M)$, and $\mathrm{Ein}\, g = \mathrm{Ein}\, \hat{g}$, $\kappa \neq -\frac{1}{n}, \Lambda \neq 0$. Moreover, suppose*

(i) M is not conformally diffeomorphic to $\mathbf{H}^n$ or $\mathbf{R} \times M_*$, where M_* is a complete $(n-1)$-dimensional manifold, if $\frac{\Lambda}{\kappa n+1} > 0$; or

(ii) M is not conformally diffeomorphic to $\mathbf{S}^n$, $\mathbf{H}^n$ or $\mathbf{R} \times M_*$, where M_* is a complete $(n-1)$-dimensional manifold, if $\frac{\Lambda}{\kappa n+1} < 0$.

Then $\varphi = 1$.

The proofs of Theorems 3.1, 3.2, 3.3 are given in Sections 3.3, 3.4 and 3.5.

Whether $\mathbf{H}^n$ or $I \times M_*$ admit nonconstant φ such that $\mathrm{Ein}\, g = \mathrm{Ein}\, \hat{g}$ is still open. It is also not clear whether $\mathbf{S}^n$ admits such a φ when $\frac{\Lambda}{\kappa n+1} < 0$.

Notation. We will use ∇ to denote the Levi-Cevita connection on (M, g). Given smooth function φ, we will denote the Hessian by $\operatorname{Hess}\varphi$, the gradient by $\operatorname{grad}\varphi$, Laplace-Beltrami by $\Delta\varphi = g^{ij}\nabla_i\nabla_j\varphi$ and the norm squared of the gradient by $|\,\mathrm{d}\varphi\,| = g^{ij}\nabla_i\varphi\nabla_j\varphi$ respectively. All four of these will always be meant with respect to g.

We will also adopt the notation that if a quantity Ω is formed with respect to g then the corresponding quantity formed with respect to $\hat{g}$ will be denoted $\hat{\Omega}$. For example, in local coordinates, the Ricci curvature of g is denoted R_{ij} and the Ricci curvature of $\hat{g}$ is denoted $\hat{R}_{ij}$.

3.2 General Formulas for Conformal Transformations

The goal of this chapter is to prove Theorems 3.1, 3.2, 3.3. We will begin by deriving the general transformation of Ein after a conformal change of metric. All the results presented in this section hold for any Riemannian manifold (M, g) with any dimension n unless explicitly stated otherwise.

Proposition 3.4. *If $\hat{g} = \varphi^{-2}g$, $\varphi \in C_+^\infty(M)$, then we have the following transformations.*

Ricci Curvature:

$$\operatorname{Ric}\hat{g} = \operatorname{Ric} g + (n-2)\frac{1}{\varphi}\operatorname{Hess}\varphi + \frac{1}{\varphi^2}\big(\varphi\Delta\varphi - (n-1)|\,\mathrm{d}\varphi\,|^2\big)g, \tag{3.2}$$

Traceless-Ricci Curvature:

$$\operatorname{Ric}^\circ\hat{g} = \operatorname{Ric}^\circ g + (n-2)\frac{1}{\varphi}\operatorname{Hess}\varphi - \frac{n-2}{n\varphi}(\Delta\varphi)g \tag{3.3}$$

Scalar Curvature:

$$\hat{S} = \varphi^2 S + 2(n-1)\varphi\Delta\varphi - n(n-1)|\,\mathrm{d}\varphi\,|^2 \tag{3.4}$$

Proof. We will begin by showing (3.2). Fix a set of local coordinates $\{x^i\}_{i=1}^n$. The components of the Ricci curvature are given by

$$R_{ij} = g^{st}R_{isjt}$$

where R_{ijkl} are the components of the (0,4)-Riemannian curvature tensor. It follows from Theorem A.2 that

$$\begin{aligned}\hat{R}_{ij} &= \varphi^2 g^{st}\left(\frac{1}{\varphi^2}R_{isjt} - \frac{1}{\varphi^4}(g \owedge T)_{isjt}\right)\\ &= R_{ij} - \frac{1}{\varphi^2}g^{st}(g \owedge T)_{isjt}\end{aligned}$$

where $T_{ij} = -\varphi\nabla_i\nabla_j\varphi + \frac{1}{2}|\,\mathrm{d}\varphi\,|^2 g_{ij}$ and $⊘$ is the Kulkarni-Nomizu product (see Appendix A for the definition). Using that

$$g^{st}T_{st} = -\varphi\Delta\varphi + \frac{n}{2}|\,\mathrm{d}\varphi\,|^2$$

gives

$$\begin{aligned} g^{st}(g \owedge T)_{isjt} &= \Big(-\varphi\Delta\varphi + \frac{n}{2}|\,\mathrm{d}\varphi\,|^2 \Big) g_{ij} + (n-2)T_{ij} \\ &= -(n-2)\varphi\nabla_i\nabla_j\varphi - \big(\varphi\Delta\varphi - (n-1)|\,\mathrm{d}\varphi\,|^2\big) g_{ij}, \end{aligned}$$

so

$$\hat{R}_{ij} = R_{ij} + (n-2)\frac{1}{\varphi}\nabla_i\nabla_j\varphi + \frac{1}{\varphi^2}\big(\varphi\Delta\varphi - (n-1)|\,\mathrm{d}\varphi\,|^2\big) g_{ij}. \tag{3.5}$$

The local coordinates were arbitrary, so (3.5) can be replaced by the global equation (3.2).

Next, we will prove (3.4). The scalar curvature S is the trace of $\operatorname{Ric} g$ with respect to g. In an arbitrary set of local coordinates $\{x^i\}_{i=1}^n$, this is expressed by $S = g^{\alpha\beta}R_{\alpha\beta}$. By (3.5), we have

$$\begin{aligned} \hat{S} &= \varphi^2 g^{\alpha\beta}\Big(R_{\alpha\beta} + (n-2)\frac{1}{\varphi}\nabla_\alpha\nabla_\alpha\varphi + \frac{1}{\varphi^2}\big(\varphi\Delta\varphi - (n-1)|\,\mathrm{d}\varphi\,|^2\big) g_{\alpha\beta} \Big) \\ &= \varphi^2 S + 2(n-1)\varphi\Delta\varphi - n(n-1)|\,\mathrm{d}\varphi\,|^2 \end{aligned}$$

as required.

Finally, (3.3) follows easily from (3.2) and (3.4). Indeed,

$$\begin{aligned} \operatorname{Ric}^\circ \hat{g} &= \operatorname{Ric}\hat{g} - \frac{1}{n}\hat{S}\hat{g} \\ &= \operatorname{Ric} g + (n-2)\frac{1}{\varphi}\operatorname{Hess}\varphi + \frac{1}{\varphi^2}\big(\varphi\Delta\varphi - (n-1)|\,\mathrm{d}\varphi\,|^2\big) g \\ &\quad - \frac{1}{n}\Big(\varphi^2 S + 2(n-1)\varphi\Delta\varphi - n(n-1)|\,\mathrm{d}\varphi\,|^2\Big)\cdot\frac{1}{\varphi^2} g \\ &= \operatorname{Ric}^\circ g + (n-2)\frac{1}{\varphi}\operatorname{Hess}\varphi - \frac{n-2}{n\varphi}(\Delta\varphi) g. \end{aligned}$$

□

Proposition 3.4 readily implies the transformation of Trace-Adjusted Ricci Tensors.

Theorem 3.5. *Let (M, g) be a Riemannian manifold. If $\hat{g} = \varphi^{-2}g$, $\varphi \in C_+^\infty(M)$, then*

$$\begin{aligned} \operatorname{Ein}\hat{g} &= \operatorname{Ein} g + (n-2)\frac{1}{\varphi}\operatorname{Hess}\varphi + \Lambda\Big(\frac{1}{\varphi^2} - 1\Big) g \\ &\quad + \frac{1}{\varphi^2}\Big((2\kappa n - 2\kappa + 1)\varphi\Delta\varphi - (n-1)(\kappa n + 1)|\,\mathrm{d}\varphi\,|^2 \Big) g \end{aligned}$$

Proof. By direct computation,

$$\begin{aligned}
\operatorname{Ein}\hat{g} &= \operatorname{Ric}\hat{g} + \kappa\hat{S}\hat{g} + \Lambda\hat{g} \\
&= \operatorname{Ric} g + (n-2)\frac{1}{\varphi}\operatorname{Hess}\varphi + \frac{1}{\varphi^2}\big(\varphi\Delta\varphi - (n-1)|\,\mathrm{d}\varphi\,|^2\big)g \\
&\qquad + \kappa\Big(\varphi^2 S + 2(n-1)\varphi\Delta\varphi - n(n-1)|\,\mathrm{d}\varphi\,|^2\Big)\cdot\frac{1}{\varphi^2}g + \frac{\Lambda}{\varphi^2}g \\
&= \operatorname{Ein} g + (n-2)\frac{1}{\varphi}\operatorname{Hess}\varphi + \Lambda\bigg(\frac{1}{\varphi^2} - 1\bigg)g \\
&\quad + \frac{1}{\varphi^2}\Big((2\kappa n - 2\kappa + 1)\varphi\Delta\varphi - (\kappa n^2 + (1-\kappa)n - 1)|\,\mathrm{d}\varphi\,|^2\Big)g \\
&= \operatorname{Ein} g + (n-2)\frac{1}{\varphi}\operatorname{Hess}\varphi + \Lambda\bigg(\frac{1}{\varphi^2} - 1\bigg)g \\
&\quad + \frac{1}{\varphi^2}\Big((2\kappa n - 2\kappa + 1)\varphi\Delta\varphi - (n-1)(\kappa n + 1)|\,\mathrm{d}\varphi\,|^2\Big)g.
\end{aligned}$$

□

3.2.1 Conformal Transformations Preserving Trace-Adjusted Ricci Tensors

We are now at a point at which we can talk about conformal transformation which preserve a trace-adjusted Ricci tensor. Indeed, it follows directly from Theorem 3.5 that if $\operatorname{Ein} g = \operatorname{Ein}\hat{g}$ then φ must satisfy a global equation relating $\operatorname{Hess}\varphi$ to the metric g. In fact, we will prove $\operatorname{Ein} g = \operatorname{Ein}\hat{g}$ implies that φ must necessarily satisfy a second such equation. Moreover, this equation is independent of the first equation if and only if $\kappa \neq -\frac{1}{n}$ which is why this case must be treated separately.

We begin with a small Lemma.

Lemma 3.6. *Let (M, g) be a Riemannian manifold. Suppose $\hat{g} = \varphi^{-2}g$, $\varphi \in C^\infty_+(M)$. If $\operatorname{Ein} g = \operatorname{Ein}\hat{g}$ then $\operatorname{Ric}^\circ g = \operatorname{Ric}^\circ \hat{g}$.*

Proof. Let $\{x^i\}_{i=1}^n$ be an arbitrary set of local coordinates and E_{ij} the components of $\operatorname{Ein} g$ in these coordinates. Since $\operatorname{Ein}\hat{g} = \operatorname{Ein} g$ it follows that in local coordinates the traceless part of $\operatorname{Ein}\hat{g}$ is given by

$$\begin{aligned}
\hat{E}_{ij} - \frac{1}{n}\big(\hat{g}^{\alpha\beta}\hat{E}_{\alpha\beta}\big)\hat{g}_{ij} &= E_{ij} - \frac{1}{n}\varphi^2\big(g^{\alpha\beta}E_{\alpha\beta}\big)\cdot\frac{1}{\varphi^2}g_{ij} \\
&= E_{ij} - \frac{1}{n}\big(g^{\alpha\beta}E_{\alpha\beta}\big)g_{ij},
\end{aligned}$$

the traceless part of $\operatorname{Ein} g$. Observe that the trace of $\operatorname{Ein} g$ with respect to the metric g is given by $(1+\kappa n)S + n\Lambda$. Hence, the traceless part of $\operatorname{Ein} g$ is given by

$$\begin{aligned}
\operatorname{Ein} g - \frac{1}{n}\big((1+\kappa n)S + n\Lambda\big)g_{ij} &= \operatorname{Ric} g + kSg + \Lambda g - \frac{1}{n}((1+\kappa n)S + n\Lambda)g \\
&= \operatorname{Ric} g - \frac{1}{n}Sg
\end{aligned}$$

which is simply $\operatorname{Ric}^\circ g$. Analogously, the traceless part of $\operatorname{Ein}\hat{g}$ equals $\operatorname{Ric}^\circ \hat{g}$. It follows that $\operatorname{Ric}^\circ \hat{g} = \operatorname{Ric}^\circ g$. $\square$

This leads to the following Theorem.

Theorem 3.7. *Let (M, g) be a Riemannian manifold. Suppose $\hat{g} = \varphi^{-2}g$, $\varphi \in C^\infty_+(M)$ and $\operatorname{Ein} g = \operatorname{Ein}\hat{g}$. Then φ satisfies*

$$\begin{aligned}(n-2)\operatorname{Hess}\varphi = {}&\frac{1}{\varphi}\Big((-2\kappa n + 2\kappa - 1)\varphi\Delta\varphi + (n-1)(\kappa n + 1)|\,\mathrm{d}\varphi\,|^2\Big)g \\ &+ \Lambda\Big(\varphi - \frac{1}{\varphi}\Big)g\end{aligned} \tag{3.6}$$

and, if $n \geqslant 3$,

$$\operatorname{Hess}\varphi = \frac{\Delta\varphi}{n}g \tag{3.7}$$

Proof. Equation (3.6) follows directly from setting $\operatorname{Ein} g = \operatorname{Ein}\hat{g}$ in Theorem 3.5. By Lemma 3.6, we also have $\operatorname{Ric}^\circ g = \operatorname{Ric}^\circ \hat{g}$. Then Proposition 3.4 implies

$$(n-2)\frac{1}{\varphi}\operatorname{Hess}\varphi - \frac{n-2}{n\varphi}(\Delta\varphi)g = 0$$

which for $n \geqslant 3$ gives (3.7). $\square$

Remark. The fact (3.7) holds only for $n \geqslant 3$ reflects the fact that in two dimensions the traceless Ricci curvature is a conformal invariant.

An important observation is (3.7) is independent of κ, Λ and n (provided $n \geqslant 3$). This indicates that understanding which manifolds admit nonconstant solutions to (3.7) are going to be critical to proving the uniqueness problem.

Another observation is that if $\kappa = -\frac{1}{n}$ then (3.6) reduces to

$$(n-2)\Big(\operatorname{Hess}\varphi - \frac{\Delta\varphi}{n}g\Big) = \Lambda\Big(\varphi - \frac{1}{\varphi}\Big)g.$$

Hence, if $n \geqslant 3$ and $\Lambda = 0$ then (3.6) and (3.7) are the same equation. Furthermore, if $n \geqslant 3$, but $\Lambda \neq 0$ then (3.6) and (3.7) imply that

$$\Lambda\Big(\varphi - \frac{1}{\varphi}\Big)g = 0$$

from which it follows that $\varphi = 1$ (since $\varphi > 0$). These observations can be combined into a single global equation which will be useful for future calculations. Though this discussion was for $n \geqslant 3$, the Theorem 3.8 below still holds for $n = 2$.

Theorem 3.8. *Suppose $\hat{g} = \varphi^{-2}g$, $\varphi \in C^\infty_+(M)$ and $\operatorname{Ein} g = \operatorname{Ein}\hat{g}$. Then*

$$(n-1)(\kappa n + 1)\Big(\frac{2}{n}\varphi\Delta\varphi - |\,\mathrm{d}\varphi\,|^2\Big) + \Lambda\Big(\frac{1}{\varphi} - \varphi\Big) = 0. \tag{3.8}$$

Proof. The case $n \geqslant 3$ follows directly from substituting (3.7) into (3.6). When $n = 2$ (3.6) becomes

$$0 = \frac{1}{\varphi}\Big(- (2\kappa+1)\varphi\Delta\varphi + (2\kappa+1)|\,\mathrm{d}\varphi\,|^2\Big)g + \Lambda\Big(1 - \frac{1}{\varphi^2}\Big)g.$$

Multiplying by $-\varphi$ implies

$$(2\kappa+1)\Big(\varphi\Delta\varphi - |\,\mathrm{d}\varphi\,|^2\Big) - \Lambda\Big(\varphi - \frac{1}{\varphi}\Big)g = 0$$

which is precisely (3.8). □

Theorem 3.8 implies some interesting corollaries for compact manifolds with and without boundary.

Corollary 3.9. *Suppose* (M, g) *is compact without boundary. If* $\hat{g} = \varphi^{-2}g$ *and* $\operatorname{Ein} g = \operatorname{Ein}\hat{g}$*,* $\kappa \neq -\frac{1}{n}, \Lambda = 0$*, then* φ *is a constant.*

Proof. Integrating Theorem 3.8 then applying integration by parts gives,

$$\begin{aligned}\int_M |\,\mathrm{d}\varphi\,|^2\,\mathrm{d}\mu &= \frac{2}{n}\int_M \varphi\Delta\varphi\,\mathrm{d}\mu \\ &= -\frac{2}{n}\int_M |\,\mathrm{d}\varphi\,|^2\,\mathrm{d}\mu\end{aligned}$$

where μ is the Riemannian volume element with respect to g. It follows that

$$\left(1 + \frac{2}{n}\right)\int_M |\,\mathrm{d}\varphi\,|^2\,\mathrm{d}\mu = 0,$$

so φ must be a constant. □

Corollary 3.9 generalises the results [22] from the Ricci curvature to all trace-adjusted Ricci tensors with $\kappa \neq -\frac{1}{n}$ and $\Lambda = 0$.

In the next corollary we consider the case M has a boundary ∂M. Let ν denote a unit normal field and H the mean curvature of ∂M with respect to g.

Corollary 3.10. *There is no conformal transformation between a compact Riemannian manifold* M *with boundary* ∂M *such that* $H \leqslant 0$*,* $\hat{H} \geqslant 0$ *and* $\operatorname{Ein} g = \operatorname{Ein}\hat{g}$ *unless* $H = \hat{H} = 0$*.*

Proof. Integrating Theorem 3.8 then applying integration by parts gives,

$$\begin{aligned}\int_M |\,\mathrm{d}\varphi\,|^2\,\mathrm{d}\mu &= \frac{2}{n}\int_M \varphi\Delta\varphi\,\mathrm{d}\mu \\ &= -\frac{2}{n}\int_M |\,\mathrm{d}\varphi\,|^2\,\mathrm{d}\mu + \frac{2}{n}\int_{\partial M} \varphi\frac{\partial\varphi}{\partial\nu}\,\mathrm{d}s\end{aligned}$$

where $\mathrm{d}s$ is the surface element of ∂M and $\frac{\partial \varphi}{\partial \nu} = g(\operatorname{grad}\varphi, \nu)$ which further implies

$$\left(1+\frac{2}{n}\right)\int_M |\,\mathrm{d}\varphi\,|^2\,\mathrm{d}\mu = \frac{2}{n}\int_{\partial M}\varphi\frac{\partial\varphi}{\partial\nu}\,\mathrm{d}s.$$

It is well known (see for example [4, Sect. 1.J]) that the mean curvature of ∂M with respect to $\hat{g}$ is given by

$$\hat{H} = \varphi H - \frac{\partial\varphi}{\partial\nu}.$$

Since $H \leqslant 0$ and $\hat{H} \geqslant 0$ it follows that $\frac{\partial\varphi}{\partial\nu} \leqslant 0$, so

$$0 \leqslant \left(1+\frac{2}{n}\right)\int_M |\,\mathrm{d}\varphi\,|^2\,\mathrm{d}\mu \leqslant 0.$$

Thus, φ must be a constant and $\hat{H} = \varphi H$ which further implies $H = \hat{H} = 0$. □

3.3 Concircular Functions

In the previous section we showed that if $\hat{g} = \varphi^{-2}g$, $\varphi \in C^\infty_+(M)$ and $\operatorname{Ein} g = \operatorname{Ein}\hat{g}$ then φ must satisfy

$$\operatorname{Hess}\varphi = \frac{\Delta\varphi}{n}g. \tag{3.9}$$

(Recall that we are assuming $n \geqslant 3$). This is surprising since Ein depends on κ and Λ, but (3.9) is independent of these constants. This indicates that functions satisfying (3.9) are important to our uniqueness problem.

Definition. A *concircular function* is a smooth function φ from M to $\mathbf{R}$ which satisfies (3.9).

We would like to know which manifolds admit nonconstant, positive concircular functions since this will tell us which manifolds can possibly admit nonhomothetic metrics with the same trace-adjusted Ricci curvature. In this section, we are going to build up some of the theory of concircular functions, ending with the result by Tashiro, [20], that the only complete, connected Riemannian manifolds with dimension $n \geqslant 3$ admitting nonconstant concircular functions are

$$\mathbf{S}^n, \mathbf{R}^n, \mathbf{H}^n, \mathbf{R}\times M_*$$

where M_* is a complete $(n-1)$-dimensional manifold with the obvious metrics.

We will begin by giving a few properties of concircular functions follow (almost) directly from the definition. Recall the following definitions.

Definition. Suppose that $\varphi : M \to \mathbf{R}$ is smooth. A *critical point* of a smooth function $\varphi : M \to \mathbf{R}$ is a point $p \in M$ at which $|\,\mathrm{d}\varphi_p\,| = 0$. If $p \in M$ is not a critical point then it is called an *ordinary point.* Moreover, a *critical value* is a point $q \in \mathbf{R}$ such that $\varphi^{-1}(\{q\})$ contains a critical point and $q \in \mathbf{R}$ is an *ordinary value* if it is not a critical value.

Proposition 3.11 (Properties of Concircular Functions, [18, Lem. 11])**.** *Let $\varphi : M \to \mathbf{R}$ be concircular. Then in an open set of M containing no critical points there holds:*

(i) Integral curves of $\operatorname{grad}\varphi$ *are geodesics.*

(ii) Along φ-hypersurfaces $|\,\mathrm{d}\varphi\,|$ is constant.

(iii) Along unit speed geodesics in the direction of $\operatorname{grad}\varphi$*, $n\varphi'' = \Delta\varphi$.*

Proof. The unit normal of φ-hypersurfaces is given by $\nu = \frac{\operatorname{grad}\varphi}{|\,\mathrm{d}\varphi\,|}$. Observe that for each $X \in \mathfrak{X}(M)$, $\operatorname{Hess}(\varphi) = \frac{\Delta\varphi}{n} g$ is equivalent to $\nabla_X \operatorname{grad}(\varphi) = \frac{\Delta\varphi}{n}$. Hence,

$$\nabla_X \nu = \frac{\nabla_X \operatorname{grad}(\varphi)}{|\,\mathrm{d}\varphi\,|} - \frac{\Delta\varphi}{n} \cdot \frac{g(X, \operatorname{grad}(\varphi))}{|\,\mathrm{d}\varphi\,|^3} \operatorname{grad}(\varphi)$$

where we used that

$$\begin{aligned} \nabla_X |\,\mathrm{d}\varphi\,| &= \frac{g(\nabla_X \operatorname{grad}(\varphi), \operatorname{grad}(\varphi))}{|\,\mathrm{d}\varphi\,|} \\ &= \frac{1}{|\,\mathrm{d}\varphi\,|} \cdot \frac{\Delta\varphi}{n} g(X, \operatorname{grad}(\varphi)). \end{aligned} \tag{3.10}$$

Thus,

$$\nabla_X \nu = \frac{\Delta\varphi}{n} \cdot \frac{1}{|\,\mathrm{d}\varphi\,|} \left(X - \frac{g(X, \operatorname{grad}(\varphi))}{|\,\mathrm{d}\varphi\,|^2} \operatorname{grad}(\varphi) \right).$$

For (i), observe that setting $X = \nu$ gives $\nabla_\nu \nu = 0$.

For (ii), it follows directly from (3.10) that if X is orthogonal to ν then $\nabla_X |\,\mathrm{d}\varphi\,| = 0$ so $|\,\mathrm{d}\varphi\,|$ is a constant on φ-hypersurfaces.

Finally, for (iii), let $\gamma(t)$ be a unit speed geodesic in the direction of $\operatorname{grad}\varphi$. By the definition of grad, along γ,

$$\frac{\mathrm{d}\varphi}{\mathrm{d}t} = g(\operatorname{grad}(\varphi), \gamma').$$

Then

$$\begin{aligned} \frac{\mathrm{d}^2\varphi}{\mathrm{d}t^2} &= g(\nabla_{\gamma'} \operatorname{grad}(\varphi), \gamma'(t)) + g(\operatorname{grad}(\varphi), \nabla_{\gamma'}\gamma') \\ &= \frac{\Delta\varphi}{n} g(\gamma', \gamma'). \end{aligned}$$

□

The next Lemma gives a new characterisation of nonconstant concircular functions φ which is essential for classifying manifolds admitting such functions. The idea behind the Lemma is that at any ordinary point p there exist local coordinates such that φ depends on only one coordinate. This has the implication that any PDE is turned in an ODE. In particular, this will give an easy way to come up with nonconstant concircular function on those manifolds which do admit them.

Lemma 3.12 ([20, Lem. 1.2]). *Let $p \in M$ be an ordinary point of a nonconstant smooth function $\varphi : M \to \mathbf{R}$. Then φ is concircular in a neighbourhood of p if and only if there exists local coordinates $\{x^i\}$ about p such that φ only depends on on x^n and the first fundamental form is given by*

$$\mathrm{d}s^2 = (\varphi')^2 \, \mathrm{d}s_*^2 + (\mathrm{d}x^n)^2 \tag{3.11}$$

where $\mathrm{d}s_*^2$ is independent of x^n and the prime denotes ordinary differentiation with respect to x^n.

Remark. 1. Lemma 3.12 for the case M is an Einstein manifold is due to Brinkmann [5, Sect. 3]. Lemma 3.12 also holds when M is a pseudo-Riemannian manifold, see [13, Sect. 12].

2. The metric (3.11) is an example of a *warped product metric*. Warped products are very common in geometry and have applications in many areas of mathematics and physics. As such there are many treatises on them; see for example [4, Sect. 9.J].

Before we give the proof of Lemma 3.12 we will first provide some motivation behind the choice of coordinates. Let q_0 be an ordinary value of φ. Then there is an open interval $I \subset \mathbf{R}$ containing q_0 such that each $q \in I$ is an ordinary value. By the Implicit Function Theorem, each $\varphi^{-1}(\{q\})$ is an $(n-1)$-dimensional manifold. We refer to these manifolds as φ-hypersurfaces. Coordinates on the φ-hypersurfaces will correspond to the coordinates $x^1, \ldots, x^{n-1}$ in Lemma 3.12. Then the final coordinate x^n corresponding to the arc length of curves that are tangential to φ-hypersurface. Such curves are referred to as φ-curves. The trick will be to choose $x^1, \ldots, x^{n-1}$ and x^n such that these coordinates are independent of the φ-hypersurfaces and φ-curves respectively. The proof of Lemma 3.12 is based off of the proof of Lemma 12 in [18].

Proof of Lemma 3.12. First suppose there exist coordinates $\{x^i\}_{i=1}^n$ as described in the statement of the Lemma. We wish to show φ is concircular. Let g_* be the metric associated with $\mathrm{d}s_*^2$, and $(\Gamma_*)_{ij}^k$ $(1 \leqslant i, j, k \leqslant n-1)$ the Christoffel symbols with respect to g_*. A prime will always denote ordinary differentiation with respect to x^n. One can calculate that in the coordinates $x^1, \dots, x^n$ the Chritoffel symbols are given by

$$\Gamma^k_{ij} = \left(\begin{array}{ccc|c} & & & \varphi''/\varphi' \\ & (\Gamma_*)^k_{ij} & & \vdots \\ & & & \varphi''/\varphi' \\ \hline \varphi''/\varphi' & \cdots & \varphi''/\varphi' & 0 \end{array}\right), \quad 1 \leqslant k \leqslant n-1$$

$$\Gamma^n_{ij} = \left(\begin{array}{ccc|c} & & & 0 \\ & -\varphi'\varphi''(g_*)_{ij} & & \vdots \\ & & & 0 \\ \hline 0 & \cdots & 0 & 0 \end{array}\right) = \left(\begin{array}{ccc|c} & & & 0 \\ & -\frac{\varphi''}{\varphi'} g_{ij} & & \vdots \\ & & & 0 \\ \hline 0 & \cdots & 0 & 0 \end{array}\right).$$

Then

$$\begin{aligned} \nabla_i \nabla_j \varphi &= \frac{\partial^2 \varphi}{\partial u^i \partial u^j} - \Gamma^n_{ij} \varphi' \\ &= \varphi'' g_{ij}. \end{aligned}$$

Hence, φ is concircular and $\varphi'' = \frac{\Delta\varphi}{n}$.

Now suppose φ is concircular and let us construct coordinates with the stated properties. Let $M_p = \{q \in M : \varphi(q) = \varphi(p)\}$. By the implicit function theorem, M_p is a smooth manifold of dimension $n-1$. Let $x^1, \dots, x^{n-1}$ be local coordinates on M_p. Via the exponential map, one can extend these coordinates to $x^1, \dots, x^n$ in a such a way that x^n-curves are geodesics with x^n as arc length and $\frac{\partial}{\partial x^n}$ are orthogonal to each x^n-hypersurface: $\{q \,|\, x^n(q) \text{ is constant}\}$. This implies that the x^n-hypersurfaces are parallel.

Next, by Proposition 3.11, $|\,\mathrm{d}\varphi\,|$ is constant along φ-hypersurfaces, so each φ-hypersurface is parallel. Since M_p equals the x^n-hypersurface containing p, it follows that the x^n-hypersurfaces coincide with the φ-hypersurfaces. This implies that φ can be regarded as a function of x^n. Hence, $\varphi = \varphi(x^n)$ and

$$\operatorname{grad} \varphi = \varphi' \frac{\partial}{\partial x^n}.$$

By constuction $g_{nn} = 1$ and $g_{in} = 0$ for all $1 \leqslant i \leqslant n-1$. All that is left to be shown is that $(\varphi(x^n))^{-2} g_{ij}$, $1 \leqslant i, j \leqslant n-1$, is independent of x^n. Indeed, for such i, j,

$$\begin{aligned} \frac{\partial g_{ij}}{\partial x^n} &= g\left(\nabla_i \frac{\partial}{\partial x^n}, \frac{\partial}{\partial x^j}\right) + g\left(\frac{\partial}{\partial x^i}, \nabla_j \frac{\partial}{\partial x^n}\right) \\ &= g\left(\nabla_i \frac{\operatorname{grad}\varphi}{\varphi'}, \frac{\partial}{\partial x^j}\right) + g\left(\frac{\partial}{\partial x^i}, \nabla_j \frac{\operatorname{grad}\varphi}{\varphi'}\right) \\ &= \frac{2}{\varphi'} \cdot \frac{\Delta\varphi}{n} g_{ij} \\ &= 2\frac{\varphi''}{\varphi'} g_{ij}. \end{aligned}$$

Thus,

$$\frac{\partial}{\partial x^n}\left(\frac{g_{ij}}{(\varphi')^2}\right) = \frac{1}{(\varphi')^2}\frac{\partial g_{ij}}{\partial x^n} - 2\varphi'' \cdot \frac{g_{ij}}{(\varphi')^3} = 0.$$

This proves that the first fundamental form can be written as (3.11) where

$$\mathrm{d}s_*^2 = (\varphi(x^n))^{-2} g_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j$$

with the summation only over $1 \leqslant i, j \leqslant n-1$. □

The importance of Lemma 3.12 is that it allows us to explicitly give examples of nonconstant concircular functions on certain manifolds. Note that Lemma 3.12 does not say anything about whether φ is positive. Let us now give some examples of nonconstant concircular functions.

Example 3.1. $M = \mathbf{R}^n$. Let $\varphi : \mathbf{R}^n \to \mathbf{R}$, $p \mapsto \frac{1}{2}|p|^2 + c$. In standard Euclidean coordinates, we can easily calculate $\nabla_i \nabla_j \varphi = \delta_{ij}$, so φ is a global concircular function. Taking $c > 0$ implies $\varphi > 0$.

Example 3.2. Let $M = \mathbf{S}^n$ and $\mathrm{d}s^2 = \mathrm{d}u^2 + \cos^2 u\,\mathrm{d}s_1^2$ as in Lemma 3.12. The corresponding concircular function is $\varphi(u) = \sin u + c$ where $c > 1$ so that $\varphi > 0$. Even though φ was defined using local coordinates, φ is defined globally due to periodicity of the sine function.

Example 3.3. Let $M = \mathbf{H}^n$ and $\mathrm{d}s^2 = \mathrm{d}u^2 + \sinh^2 u\,\mathrm{d}s_1^2$ as in Lemma 3.12.. The corresponding concircular function if $\varphi(u) = \cosh u + c$ where $c > -1$ so that $\varphi > 0$.

Example 3.4. Let $M = \mathbf{R} \times M_*$ where M_* is an arbitrary complete $n-1$-dimensional manifold. Let $\varphi : \mathbf{R} \times M_* \to \mathbf{R}$ be defined by $(u, p) \mapsto \arctan u + \frac{\pi}{2}$. By Lemma 3.12, φ is a globally defined concircular. Indeed, any smooth function $\varphi : \mathbf{R} \to \mathbf{R}$ which satisfies

(i) $\varphi > 0$ and $\varphi' > 0$;

(ii) $\int_0^\infty \frac{1}{\varphi(u)}\,\mathrm{d}u$; and

(iii) $\lim_{u\to-\infty} \varphi(u) = 0$

induces a globally defined conircular function of $\mathbf{R} \times M_*$ via $(u, p) \mapsto \varphi(u)$, see [18, Sect. E, Example 3] for more details.

Examples 1-4 where found in [18, Sect. E].

Recall that the goal of this section is classify those complete, connected Riemannian manifolds which admit nonconstant concircular functions. So far we have shown that $\mathbf{S}^n$, $\mathbf{R}^n$, $\mathbf{H}^n$ and $\mathbf{R} \times M_*$ admit such a function. In fact, it has been proven that these are the *only* examples of complete, connected Riemannian manifolds admitting nonconstant concircular functions. The precise statement of this result is the subject of Theorem 3.13. The proof is beyond the scope of this thesis, but relies on Lemma 3.12 and the fact that critical points of concircular functions are isolated (see [20, Sect. 2]).

Theorem 3.13 ([20, Thm 1])**.** *Let (M, g) be a complete, connected Riemannian manifold admitting a nonconstant concircular function φ. Then the number of critical points of φ is $N \leqslant 2$ and M is conformally diffeomorphic to*

(i) the sphere $(\mathbf{S}^n, g_1)$ if $N = 2$;

(ii) Euclidean space $(\mathbf{R}^n, g_0)$ or Hyperbolic space $(\mathbf{H}^n, g_{-1})$ if $N = 1$; or

(iii) the Riemannian product $\mathbf{R} \times M_*$ if $N = 0$, where M_* is a complete $(n-1)$-manifold.

Remark. The formulation of Theorem 3.13 given here was made in [18, Lem. 12].

Observe that Theorem 3.13 directly implies Theorem 3.2. Indeed, when $\kappa = -\frac{1}{n}$ and $\Lambda = 0$, Theorem 3.5 implies that $\operatorname{Ein} g = \operatorname{Ein} \hat{g}$, where $\hat{g} = \varphi^{-2} g$, $\varphi \in C^\infty_+(M)$ if and only if φ is a positive concircular function.

We will make use of the following corollary for the proof of Theorem 3.3.

Corollary 3.14. *Let (M, g) be a compact, connected Riemannian manifold admitting a nonconstant concircular function. Then (M, g) is conformally diffeomorphic to $(\mathbf{S}^n, g_1)$.*

3.4 Proof of Theorem 3.1

At this point we have built up enough of the theory of concircular functions to prove Theorem 3.1.

Proof of Theorem 3.1. Suppose, for the sake of contradiction, that $\hat{g} = \frac{1}{\varphi^2} g$, $\varphi \in C^\infty_+(M,$ $\operatorname{Ein} g = \operatorname{Ein} \hat{g}$ and φ is not a constant. It follows from Theorem 3.5 that φ is a nonconstant concircular scalar field. By Theorem 3.8, φ satisfies

$$0 = \frac{2}{n} \varphi \Delta \varphi - |\, \mathrm{d}\varphi \,|^2.$$

Moreover, by Proposition 3.11, along a unit speed geodesic in the direction of $\operatorname{grad}(\varphi)$,

$$0 = 2\varphi\varphi'' - (\varphi')^2. \tag{3.12}$$

Differentiating (3.12) implies $\varphi\varphi''' = 0$ so $\varphi(t) = at^2 + bt + c$ for some $a, b, c \in \mathbf{R}$. Substituting this back into (3.12) implies

$$\begin{aligned} 0 &= 2(at^2 + bt + c)(2a) - (2a + b)^2 \\ &= 4ac - b^2. \end{aligned}$$

Hence, $b^2 = 4ac$. We must have $a \neq 0$ since if $a = 0$ then $b = 0$ which implies φ is a constant. Then, by the quadratic formula, φ is zero when $t = -\frac{b}{2a}$. This contradicts the assumption $\varphi > 0$. $\square$

3.5 Proof of Theorem 3.3

In this section, we will prove Theorem 3.3. The inclusion of the cosmological constant makes the problem significantly more challenging. Indeed, the integration by parts method used in the proof of Corollary 3.9 and the concircular functions method used in the proof of Theorem 3.3 no longer applies. Observe, however, by Theorem 3.5 and Theorem 3.8, proving Theorem 3.3 on a M amounts to proving there are no nonconstant concircular functions $\varphi \in C_+^\infty(M)$ which satisfy (3.8), restated here for the convenience of the reader:

$$(n-1)(\kappa n+1)\left(\frac{2}{n}\varphi\Delta\varphi - |\,\mathrm{d}\varphi\,|^2\right) + \Lambda\left(\frac{1}{\varphi} - \varphi\right) = 0.$$

Moreover, by Theorem 3.13, the only manifolds admitting nonconstant concircular functions are $\mathbf{S}^n, \mathbf{R}^n, \mathbf{H}^n, \mathbf{R} \times M_*$ where M_* is a complete $(n-1)$-dimensional manifold, so there are only four cases to check. We have managed to prove that there are no positive concircular functions satisfying (3.8) on $\mathbf{S}^n$ provided $\frac{\Lambda}{\kappa n+1} > 0$ or $\mathbf{R}^n$ with no constraint on κ or Λ. In the case of $\mathbf{S}^n$, we use a maximum principle type argument, and for $\mathbf{R}^n$ we compute all concircular functions and show that none of these satisfy (3.8). The question on $\mathbf{H}^n$ and $\mathbf{R} \times M_*$ is still open, but one could conceivably apply the method for proving the case $M = \mathbf{R}^n$ to (hopefully) obtain the same result.

Theorem 3.15. *Consider* $(\mathbf{S}^n, g_1)$, $n \geqslant 3$. *Suppose* $\hat{g} = \varphi^{-2}g$, $\varphi \in C_+^\infty(M)$, *and* $\operatorname{Ein} g = \operatorname{Ein} \hat{g}$, $\kappa \neq -\frac{1}{n}, \Lambda \neq 0$ *such that* $\frac{\Lambda}{\kappa n+1} > 0$. *Then* $\varphi = 1$.

Proof. We proceed by a maximum principle type argument. By Theorem 3.8, if $\hat{g} = \frac{1}{\varphi^2}g$ and $\operatorname{Ein}(\hat{g}) = \operatorname{Ein}(g)$ $(\kappa \neq -\frac{1}{n})$ then

$$k\left(\frac{1}{\varphi} - \varphi\right) = -\frac{2}{n}\varphi\Delta\varphi + |\,\mathrm{d}\varphi\,|^2, \quad k = \frac{\Lambda}{(n-1)(\kappa n+1)}. \tag{3.13}$$

By assumption $k > 0$. Since $\mathbf{S}^n$ is compact, φ attains its maximum at a point $p \in M$. At p, $\Delta\varphi \leqslant 0$ and $|\,\mathrm{d}\varphi\,| = 0$, so (3.13) implies $\varphi \leqslant \frac{1}{\varphi}$. This implies that at p, $\varphi \leq 1$ and, since φ attains its maximum at p, $\varphi \leqslant 1$ on $\mathbf{S}^n$. Likewise, φ attains its minimum at some point $q \in \mathbf{S}^n$. At q, $\Delta\varphi \geqslant 0$ and $|\,\mathrm{d}\varphi\,| = 0$, so (3.13) implies $\varphi \geqslant \frac{1}{\varphi}$. By a similar argument as before, $\varphi \geqslant 1$ on $\mathbf{S}^n$. Thus, $\varphi = 1$.

□

Next we consider $(\mathbf{R}^n, g_0)$ where g_0 is the standard Euclidean metric.

Theorem 3.16. *Consider* $(\mathbf{R}^n, g_0)$, $n \geqslant 3$. *If* $\hat{g} = \varphi^{-2}g_0$, $\varphi \in C_+^\infty(M)$, *and* $\operatorname{Ein} g = \operatorname{Ein} \hat{g}$, $\kappa \neq -\frac{1}{n}, \Lambda \neq 0$ *then* $\varphi = 1$.

To prove Theorem 3.16, we use the following Lemma. One should compare Lemma 3.17 with Example 1 in Section 3.3.

Lemma 3.17. *The only concircular functions on $\mathbf{R}^n$ are those of the form $\varphi(p) = a|p|^2 + b \cdot p + c$ where $a, c \in \mathbf{R}$, $b \in \mathbf{R}^n$ and $|\cdot|$, "$\cdot$" denote the Euclidean norm and dot product on $\mathbf{R}^n$ respectively.*

Proof. Suppose that $\varphi : \mathbf{R}^n \to \mathbf{R}$ is a concircular function. Let $\{x^i\}_{i=1}^n$ by the usual coordinates on $\mathbf{R}^n$ (i.e. if $p = (p^1, \ldots, p^n) \in \mathbf{R}^n$ then $x^i(p) = p^i$) and identify $\frac{\partial}{\partial x^i}$ with e_i the i-th standard basis vector in $\mathbf{R}^n$. For this proof, we will not use Einstein summation convention to avoid possible confusion. We have $\nabla_i \nabla_j \varphi = \frac{\partial^2 \varphi}{\partial x_i \partial x_j}$, so, since φ is concircular, $\frac{\partial^2 \varphi}{\partial x_i \partial x_j} = 0$ when $i \neq j$. This implies

$$\varphi(p) = \sum_{i=1}^n F_i(p^i) \tag{3.14}$$

for some functions $F_i : \mathbf{R} \to \mathbf{R}$. Indeed, $\frac{\partial^2 \varphi}{\partial x_i \partial x_j} = 0$ implies $\frac{\partial \varphi}{\partial x_j}$ is independent of p^i for all $i \neq j$. Hence, $\frac{\partial \varphi}{\partial x_j} = f_j(p^j)$ for some function $f_j : \mathbf{R} \to \mathbf{R}$. This hold for all j, however, so

$$\varphi(p) = F_j(p^j) + G(p^1, \ldots, p^{j-1}, p^{j+1}, \ldots, p^n)$$

where $F_j(z) = \int_0^z f_j(s)\, \mathrm{d}s$ and $G : \mathbf{R}^{n-1} \to \mathbf{R}$. One can then inductively argue that this implies (3.14). It follows that

$$\Delta \varphi = \sum_{i=1}^n F_i''(p^i).$$

Next, since $\frac{\partial^2 \varphi}{\partial x_i^2} = \frac{\Delta \varphi}{n}$ we have

$$F_i''(p^i) = \frac{1}{n} \sum_{i=1}^n F_i''(p^i).$$

This implies that $F_i''(p^i) = F_j''(p^j)$ for all i, j so each F_i'' is a constant $a \in \mathbf{R}$. Thus, each F_i is a quadratic and so

$$\varphi(p) = a|p|^2 + b \cdot p + c$$

for some $b \in \mathbf{R}^n$, $c \in \mathbf{R}$. $\square$

Remark. If $\varphi : \mathbf{R}^n \to \mathbf{R}$ is of the form $\varphi(p) = a|p|^2 + b \cdot p + c$ then $\varphi > 0$ if and only if $a > 0$ and $b_i^2 < \frac{2ac}{n}$ for $i = 1, \ldots, n$ or $a = 0, b = 0$ and $c > 0$. Indeed,

$$\varphi(p) = \sum_{i=1}^n \left(a(p^i)^2 + b^i p^i + \frac{c}{n} \right).$$

Each term in this summation is independent of the others and, if $a \neq 0$, is strictly positive if and only if $a > 0$ and the roots of the quadratic $a(p^i)^2 + b^i p^i + \frac{c}{n}$ are complex. If $a = 0$ then each term is linear and has a root if and only if $b_i \neq 0$. Hence, in this case we need $a = 0, b = 0$ and $c > 0$.

Proof of Theorem 3.16. Let φ be a concircular function on $\mathbf{R}^n$. By Theorem 3.8, φ also satisfies

$$(n-1)(\kappa n+1)\varphi\left(\frac{2}{n}\varphi\Delta\varphi - |\,\mathrm{d}\varphi\,|^2\right) + \Lambda\big(1-\varphi^2\big) = 0.$$

Lemma 3.17 implies $\varphi(p) = a|\,p\,|^2 + b\cdot p + c$ for some $a, c \in \mathbf{R}$, $b \in \mathbf{R}^n$, so substituting this into the left hand side of the above equation gives that the coefficient on the term $|p|^4$ is Λa^2. This must be zero, so $a = 0$. By the remark, however, each term $b_i = 0$ which implies $\varphi(p) = c > 0$. However, the only constant solution of (3.8) is $c = 1$, which proves the result. $\square$

Chapter 4

Cross Curvature in a Conformal Class of Metrics

4.1 Cross Curvature

In [8], Chow and Hamilton defined a new geometric flow on 3-manifolds called the Cross Curvature Flow. This flow deformed metrics with curvature of one sign via a symmetric (0,2)-tensor which Chow and Hamiliton called the *cross curvature*. The goal of Cross Curvature Flow was to produce a flow proof of the Hyperbolisation Conjecture which states that every closed 3-manifold with negative sectional curvature admits a hyperbolic metric. Here, a closed manifold is a compact manifold without boundary and a hyperbolic metric is a metric with constant sectional curvature equal to -1. Hamilton and Chow conjectured that Cross Curvature flow deforms arbitrary negatively curved metrics to a hyperbolic metric and gave evidence in [8] to support this claim. The Hyperbolisation Conjecture has since been resolved as a consequence of the resolution of the Geometrisation Conjecture by Perelman in his proof of the Poincaré Conjecture. However, the conjecture that Cross Curvature flow deforms arbitrary negatively curved metrics to a hyperbolic metric is still open. For this reason there is still some interest in the Cross Curvature tensor. In this chapter, we study the Cross Curvature in a conformal class of metrics. This is significantly more challenging than the study of the Ricci Curvature due to the fully nonlinear nature of the Cross Curvature and, as far as the author could find, has not been done anywhere in the literature.

Throughout this chapter (M^3, g) will denote a smooth Riemannian manifold of dimension 3; $\varphi : M \to \mathbf{R}$ a smooth, positive function; and $\hat{g} = \frac{1}{\varphi^2} g$ a metric conformal to g. In arbitrary choice of coordinates, the *Einstein tensor* is defined by

$$E_{ij} = R_{ij} - \frac{1}{2} S g_{ij}$$

where R_{ij} and S are the components of the Ricci and Scalar curvature of g respectively.

Note that this notation is different to that in Chapter 3 where E_{ij} denoted the components of a trace-adjusted Ricci tensor.

Definition ([8]). In an arbitrary choice of coordinates, the *Cross Curvature tensor* is the symmetric (0,2)-tensor defined by

$$X_{ij} = (\det E)V_{ij}$$

where V_{ij} is the inverse of $E^{ij} = g^{i\alpha}g^{j\beta}E_{\alpha\beta}$ and $\det E = \det(g^{i\alpha}E_{\beta j})$.

Remark. This definition makes sense for $n \geqslant 3$. However, we will only be interested in the case $n = 3$ since this will allow us to make use of a number of formulas only available in three dimensions.

Since the Cross Curvature is proportional to the inverse of the Einstein tensor, it is not always guaranteed to exist. The proposition characterises when the Cross Curvature exists.

Proposition 4.1 ([8]). *The Cross Curvature of (M^3, g) exists if and only if the sectional curvatures of (M^3, g) are all nonvanishing.*

Proof. Fix an arbitrary point $p \in M$ and (M^3, g) has strictly negative sectional curvatures (the result is analogous if the sectional curvature are strictly positive). Since E_{ij} is symmetric, there exists coordinates such that E_{ij} is diagonal and $g_{ij} = \delta_{ij}$ at p. In these coordinates R_{ij} is diagonal and so is X_{ij}. Since M is 3 dimensional, $R_{ijk\ell}$ has 6 independent components:

$$\begin{aligned} a &:= R_{1212},\ b := R_{1313},\ c := R_{2323}, \\ d &:= R_{1213},\ e := R_{2123},\ f := R_{3132}. \end{aligned}$$

Observe that a, b and c are precisely the sectional curvatures at p. Then, using that $g_{ij} = \delta_{ij}$,

$$\begin{aligned} R_{ij} &= R^{\alpha}{}_{i\alpha j} \\ &= \delta^{\alpha\beta}R_{\beta i\alpha j} \\ &= R_{1i1j} + R_{2i2j} + R_{3i3j} \\ &= \begin{pmatrix} a+b & f & e \\ f & a+c & d \\ e & d & b+c \end{pmatrix}. \end{aligned}$$

Since R_{ij} is diagonal, $d = e = f = 0$. Hence,

$$
\begin{aligned}
E_{ij} &= \begin{pmatrix} a+b & 0 & 0 \\ 0 & a+c & 0 \\ 0 & 0 & b+c \end{pmatrix} - \frac{1}{2}(2a+2b+2c)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} -c & 0 & 0 \\ 0 & -b & 0 \\ 0 & 0 & -a \end{pmatrix}
\end{aligned}
$$

and

$$
E^{ij} = \begin{pmatrix} -c & 0 & 0 \\ 0 & -b & 0 \\ 0 & 0 & -a \end{pmatrix}
$$

Thus, $\det E = -abc$, which implies the result. □

Remark. Using the same coordinates as in Proposition 4.1, one can see that

$$
V_{ij} = \begin{pmatrix} -\frac{1}{c} & 0 & 0 \\ 0 & -\frac{1}{b} & 0 \\ 0 & 0 & -\frac{1}{a} \end{pmatrix}
$$

and

$$
X_{ij} = \begin{pmatrix} ab & 0 & 0 \\ 0 & ac & 0 \\ 0 & 0 & bc \end{pmatrix}
$$

assuming $a, b, c \neq 0$. Hence, the eigenvalues of the Cross Curvature are the product of two distinct sectional curvatures which motivates the name.

Corollary 4.2. *Suppose (M^3, g) has positive sectional curvature (resp. negative sectional curvature). Then X_{ij} exists.*

4.2 Conformal Change of Metric

Let (M^3, g) be a 3 dimensional Riemannian manifold with positive sectional curvature. The purpose of this section is to calculate the Cross Curvature for a metric $\hat{g} = \frac{1}{\varphi^2} g$. Moreover, we calculate the trace of Cross Curvature and its traceless part with respect to $\hat{g}$. Finally, we consider the case that (M, g) is Einstein which simplifies all of the previous formulas.

Recall that if $\hat{g} = \frac{1}{\varphi^2} g$ for some $\varphi \in C^\infty_+(M)$ then when $n = 3$

$$
\hat{R}_{ij} = R_{ij} + \frac{1}{\varphi}\nabla_i \nabla_j \varphi + \frac{1}{\varphi^2}\left(\varphi \Delta \varphi - 2|\,\mathrm{d}\varphi\,|^2\right) g_{ij}
$$

and

$$\hat{S} = \varphi^2 S + 4\varphi\Delta\varphi - 6|\,\mathrm{d}\varphi\,|^2$$

as proven in Proposition 3.4. It follows that

$$\hat{E}^{ij} = \varphi^4 E^{ij} + \varphi^3 \nabla^i \nabla^j \varphi - \varphi^2(\varphi\Delta\varphi - |\,\mathrm{d}\varphi\,|^2)g^{ij}.$$

In order to calculate $\hat{X}_{ij}$ one would have to first find the inverse of $\hat{E}^{ij}$, $\hat{V}_{ij}$, which, in general, would be very challenging. However, we can make use of the fact that M is 3 dimensional to find a simpler expression for X_{ij}.

Lemma 4.3 ([7])**.** *Suppose (M^3, g) be a 3 dimensional Riemannian manifold with non-vanishing sectional curvature. There holds*

$$X_{ij} = -\frac{1}{2} E^{\alpha\beta} R_{i\alpha j\beta}.$$

Proof. Using the same coordinates as in Proposition 4.1, we have at each point

$$\begin{aligned} E^{\alpha\beta} R_{i\alpha j\beta} &= -cR_{i1j1} - bR_{i2j2} - aR_{i3j3} \\ &= -c\begin{pmatrix} 0 & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} - b\begin{pmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & b \end{pmatrix} - a\begin{pmatrix} b & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &= -2X_{ij}. \end{aligned}$$

□

Lemma 4.3 gives a formula for the Cross Curvature which is much more conducive to a conformal change of metric.

Before we use Lemma 4.3 to calculate $\hat{X}_{ij}$ it is worthwhile to mention that $\hat{X}_{ij}$ may not exist even if X_{ij} does. Indeed, Cross Curvature exists on the Hyperbolic plane which is conformal to $\mathbf{R}_+$ with Euclidean metric, but the Cross Curvature of a Euclidean metric is undefined. To this end, we will always assume in the forthcoming calculations that $\hat{X}_{ij}$ exists.

Before we calculate $\hat{X}_{ij}$, the following Theorem will be helpful for simplifying expressions. Let $\owedge$ denote the Kulkarni-Nomizu product defined in Appendix A.

Theorem 4.4 (Ricci Decomposition)**.** *Let (M^n, g) be a Riemannian manifold with Riemannian curvature R_{ijkl}. Then*

$$R_{ijkl} = \frac{1}{n-2}(\mathrm{Ric}^\circ \owedge g)_{ijkl} + \frac{S}{2n(n-1)}(g \owedge g)_{ijkl} + W_{ijkl}$$

where $\mathrm{Ric}^\circ{}_{ij} = R_{ij} - (S/n)g_{ij}$ *denotes the traceless Ricci tensor and* W_{ijkl} *is called the* Weyl tensor*. Moreover, when* $n = 3$*,* $W_{ijkl} = 0$*, so*

$$\begin{aligned} R_{ijkl} &= (\mathrm{Ric}^\circ \owedge g)_{ijkl} + \frac{1}{12}S(g \owedge g)_{ijkl} \\ &= (\mathrm{Ric} \owedge g)_{i\alpha j\beta} - \frac{1}{4}S(g \owedge g)_{i\alpha j\beta}. \end{aligned}$$

Remark. The statement of the theorem is in fact vacuously true since the Weyl tensor is defined as:

$$W_{ijkl} = R_{ijkl} - E_{ijkl} - S_{ijkl}$$

where

$$E_{ijkl} = \frac{1}{n-2}(\mathrm{Ric}^\circ \owedge g)_{ijkl} \qquad S_{ijkl} = \frac{S}{2n(n-1)}(g \owedge g)_{ijkl}.$$

The significance of Theorem 4.4, however, is that E_{ijkl}, S_{ijkl} and W_{ijkl} are orthogonal in the sense $E_{ijkl}S^{ijkl} = E_{ijkl}W^{ijkl} = S_{ijkl}W^{ijkl} = 0$. This is not relevant to the proof of Theorem 4.5, but is interesting and very useful in other contexts.

Theorem 4.5. *Suppose* (M^3, g) *be a 3 dimensional Riemannian manifold with nonvanishing sectional curvature,* $\hat{g} = \frac{1}{\varphi^2}g$*,* $\varphi \in C^\infty_+(M)$*, and* $\hat{X}_{ij}$ *exists. Then*

$$\begin{aligned} \hat{X}_{ij} &= \varphi^2 X_{ij} + \varphi g^{\alpha\beta}\big(R_{i\alpha}\nabla_\beta\nabla_j\varphi + R_{\beta j}\nabla_\alpha\nabla_i\varphi\big) \\ &\quad + g^{\alpha\beta}(\nabla_\alpha\nabla_i\varphi)(\nabla_\beta\nabla_j\varphi) - |\,\mathrm{d}\varphi\,|^2 R_{ij} - \left(\frac{1}{2}\varphi S + \frac{|\,\mathrm{d}\varphi\,|^2}{\varphi}\right)\nabla_i\nabla_j\varphi \\ &\quad + \left(- \varphi R^{\alpha\beta}\nabla_\alpha\nabla_\beta\varphi + \frac{1}{2}\varphi S\Delta\varphi - \frac{1}{2}|\,\mathrm{Hess}\,\varphi\,|^2 + \frac{1}{2}(\Delta\varphi)^2 - \frac{|\,\mathrm{d}\varphi\,|^2\Delta\varphi}{\varphi} + \frac{|\,\mathrm{d}\varphi\,|^4}{\varphi^2}\right) g_{ij} \end{aligned} \tag{4.1}$$

where

$$|\,\mathrm{Hess}\,\varphi\,|^2 = g^{\alpha i}g^{\beta j}(\nabla_i\nabla_j\varphi)(\nabla_\alpha\nabla_\beta\varphi).$$

Proof. By Theorem (A.2),

$$\hat{R}_{i\alpha j\beta} = \frac{1}{\varphi^2}R_{i\alpha j\beta} - \frac{1}{\varphi}(\tilde{T} \owedge g)_{i\alpha j\beta} \tag{4.2}$$

where $\tilde{T}_{ij} = -\varphi\nabla_i\nabla_j\varphi + \frac{1}{2}|\,\mathrm{d}\varphi\,|^2 g_{ij}$ and by Theorem (),

$$\hat{E}^{\alpha\beta} = \varphi^4 E^{\alpha\beta} + \varphi^3\nabla^\alpha\nabla^\beta\varphi - \varphi^2\big(\varphi\Delta\varphi - |\,\mathrm{d}\varphi\,|^2\big)g^{\alpha\beta}.$$

Hence,

$$\begin{aligned} \hat{X}_{ij} &= -\frac{1}{2}\left(\varphi^4 E^{\alpha\beta} + \varphi^3\nabla^\alpha\nabla^\beta\varphi - \varphi^2\big(\varphi\Delta\varphi - |\,\mathrm{d}\varphi\,|^2\big)g^{\alpha\beta}\right)\left(\frac{1}{\varphi^2}R_{i\alpha j\beta} - \frac{1}{\varphi}(\tilde{T} \owedge g)_{i\alpha j\beta}\right) \\ &= \varphi^2 X_{ij} - \frac{1}{2}\varphi(\nabla^\alpha\nabla^\beta\varphi)R_{i\alpha j\beta} + \frac{1}{2}\big(\varphi\Delta\varphi - |\,\mathrm{d}\varphi\,|^2\big)R_{ij} + \frac{1}{2}E^{\alpha\beta}(\tilde{T} \owedge g)_{i\alpha j\beta} \\ &\quad + \frac{1}{2\varphi}(\nabla^\alpha\nabla^\beta\varphi)(\tilde{T} \owedge g)_{i\alpha j\beta} - \frac{1}{2\varphi^2}\big(\varphi\Delta\varphi - |\,\mathrm{d}\varphi\,|^2\big)g^{\alpha\beta}(\tilde{T} \owedge g)_{i\alpha j\beta}. \end{aligned} \tag{4.3}$$

We begin by simplifying $(\nabla^\alpha\nabla^\beta\varphi)R_{i\alpha j\beta}$. By Theorem 4.4

$$R_{i\alpha j\beta} = (\mathrm{Ric} \owedge g)_{i\alpha j\beta} - \frac{1}{4}S(g \owedge g)_{i\alpha j\beta},$$

so

$$(\nabla^\alpha\nabla^\beta\varphi)R_{i\alpha j\beta} = (\nabla^\alpha\nabla^\beta\varphi)(\mathrm{Ric} \owedge g)_{i\alpha j\beta} - \frac{1}{4}S(\nabla^\alpha\nabla^\beta\varphi)(g \owedge g)_{i\alpha j\beta}.$$

Expanding the Kulkarni–Nomizu products give

$$\begin{aligned}(\nabla^\alpha\nabla^\beta\varphi)(\mathrm{Ric} \owedge g)_{i\alpha j\beta} &= (\nabla^\alpha\nabla^\beta\varphi)(R_{ij}g_{\alpha\beta} + R_\alpha g_{ij} - R_{i\beta}g_{\alpha j} - R_{\alpha j}g_{i\beta}) \\ &= (\Delta\varphi)R_{ij} + R^{\alpha\beta}(\nabla_\alpha\nabla_\beta\varphi)g_{ij} \\ &\quad - g^{\alpha\beta}(R_{i\alpha}\nabla_\beta\nabla_j\varphi + R_{\beta j}\nabla_\alpha\nabla_i\varphi)\end{aligned}$$

and

$$\begin{aligned}(\nabla^\alpha\nabla^\beta\varphi)(g \owedge g)_{i\alpha j\beta} &= 2(\nabla^\alpha\nabla^\beta\varphi)(g_{ij}g_{\alpha\beta} - g_{i\beta}g_{\alpha j}) \\ &= 2(\Delta\varphi)g_{ij} - 2\nabla_i\nabla_j\varphi. \end{aligned} \tag{4.4}$$

It follows

$$\begin{aligned}(\nabla^\alpha\nabla^\beta\varphi)R_{i\alpha j\beta} &= -g^{\alpha\beta}(R_{i\alpha}\nabla_\beta\nabla_j\varphi + R_{\beta j}\nabla_\alpha\nabla_i\varphi) + (\Delta\varphi)R_{ij} \\ &\quad + \frac{1}{2}S\nabla_i\nabla_j\varphi + \left(R^{\alpha\beta}\nabla_\alpha\nabla_\beta\varphi - \frac{1}{2}S\Delta\varphi\right)g_{ij}. \end{aligned} \tag{4.5}$$

Now we calculate $E^{\alpha\beta}(\tilde{T} \owedge g)_{i\alpha j\beta}$. Observe

$$\begin{aligned}E^{\alpha\beta}(\tilde{T} \owedge g)_{i\alpha j\beta} &= \left(R^{\alpha\beta} - \frac{1}{2}Sg^{\alpha\beta}\right) \\ &\quad \left(-\varphi(\mathrm{Hess}\,\varphi \owedge g)_{i\alpha j\beta} + \frac{1}{2}|\,\mathrm{d}\varphi\,|^2(g \owedge g)_{i\alpha j\beta}\right) \\ &= -\varphi R^{\alpha\beta}(\mathrm{Hess}\,\varphi \owedge g)_{i\alpha j\beta} + \frac{1}{2}|\,\mathrm{d}\varphi\,|^2 R^{\alpha\beta}(g \owedge g)_{i\alpha j\beta} \\ &\quad + \frac{1}{2}\varphi S g^{\alpha\beta}(\mathrm{Hess}\,\varphi \owedge g)_{i\alpha j\beta} - \frac{1}{4}S|\,\mathrm{d}\varphi\,|^2 g^{\alpha\beta}(g \owedge g)_{i\alpha j\beta}.\end{aligned}$$

Then

$$\begin{aligned}R^{\alpha\beta}(\mathrm{Hess}\,\varphi \owedge g)_{i\alpha j\beta} &= R^{\alpha\beta}(g_{ij}\nabla_\alpha\nabla_\beta\varphi + g_{\alpha\beta}\nabla_i\nabla_j\varphi - g_{i\beta}\nabla_\alpha\nabla_j\varphi - g_{\alpha j}\nabla_i\nabla_\beta\varphi) \\ &= -g^{\alpha\beta}(R_{i\alpha}\nabla_\beta\nabla_j\varphi + R_{\beta j}\nabla_\alpha\nabla_i\varphi) + S\nabla_i\nabla_j\varphi \\ &\quad + R^{\alpha\beta}(\nabla_\alpha\nabla_\beta\varphi)g_{ij},\end{aligned}$$

$$\begin{aligned}R^{\alpha\beta}(g \owedge g)_{i\alpha j\beta} &= 2R^{\alpha\beta}(g_{ij}g_{\alpha\beta} - g_{i\beta}g_{\alpha j}) \\ &= 2Sg_{ij} - 2R_{ij},\end{aligned}$$

$$
\begin{aligned}
g^{\alpha\beta}(\operatorname{Hess}\varphi \owedge g)_{i\alpha j\beta} &= g^{\alpha\beta}(g_{ij}\nabla_\alpha\nabla_\beta\varphi + g_{\alpha\beta}\nabla_i\nabla_j\varphi - g_{i\beta}\nabla_\alpha\nabla_j\varphi - g_{\alpha j}\nabla_i\nabla_\beta\varphi) \\
&= (\Delta\varphi)g_{ij} + \nabla_i\nabla_j\varphi
\end{aligned} \tag{4.6}
$$

and

$$
\begin{aligned}
g^{\alpha\beta}(g \owedge g)_{i\alpha j\beta} &= 2g^{\alpha\beta}(g_{ij}g_{\alpha\beta} - g_{i\beta}g_{\alpha j}) \\
&= 4g_{ij}.
\end{aligned} \tag{4.7}
$$

Hence,

$$
\begin{aligned}
E^{\alpha\beta}(\tilde{T} \owedge g)_{i\alpha j\beta} &= \varphi g^{\alpha\beta}(R_{i\alpha}\nabla_\beta\nabla_j\varphi + R_{\beta j}\nabla_\alpha\nabla_i\varphi) - \varphi S\nabla_i\nabla_j\varphi \\
&\quad - \varphi R^{\alpha\beta}(\nabla_\alpha\nabla_\beta\varphi)g_{ij} + |\,\mathrm{d}\varphi\,|^2 S g_{ij} - |\,\mathrm{d}\varphi\,|^2 R_{ij} \\
&\quad + \frac{1}{2}\varphi S(\Delta\varphi)g_{ij} + \frac{1}{2}\varphi S\nabla_i\nabla_j\varphi - |\,\mathrm{d}\varphi\,|^2 S g_{ij} \\
&= \varphi g^{\alpha\beta}(R_{i\alpha}\nabla_\beta\nabla_j\varphi + R_{\beta j}\nabla_\alpha\nabla_i\varphi) - |\,\mathrm{d}\varphi\,|^2 R_{ij} - \frac{1}{2}\varphi S\nabla_i\nabla_j\varphi \\
&\quad + \left(\frac{1}{2}\varphi S\Delta\varphi - \varphi R^{\alpha\beta}\nabla_\alpha\nabla_\beta\varphi\right)g_{ij}.
\end{aligned} \tag{4.8}
$$

Next, we simplify

$$
(\nabla^\alpha\nabla^\beta\varphi)(\tilde{T} \owedge g)_{i\alpha j\beta} = -\varphi(\nabla^\alpha\nabla^\beta\varphi)(\operatorname{Hess}\varphi \owedge g)_{i\alpha j\beta} + \frac{1}{2}|\,\mathrm{d}\varphi\,|^2(\nabla^\alpha\nabla^\beta\varphi)(g \owedge g)_{i\alpha j\beta}.
$$

We have

$$
\begin{aligned}
(\nabla^\alpha\nabla^\beta\varphi)(\operatorname{Hess}\varphi \owedge g)_{i\alpha j\beta} &= (\nabla^\alpha\nabla^\beta\varphi)(g_{ij}\nabla_\alpha\nabla_\beta\varphi + g_{\alpha\beta}\nabla_i\nabla_j\varphi - g_{i\beta}\nabla_\alpha\nabla_j\varphi - g_{\alpha j}\nabla_i\nabla_\beta\varphi) \\
&= |\operatorname{Hess}\varphi\,|^2 g_{ij} + (\Delta\varphi)\nabla_i\nabla_j\varphi - 2g^{\alpha\beta}(\nabla_\alpha\nabla_i\varphi)(\nabla_\beta\nabla_j\varphi),
\end{aligned}
$$

so, using (4.4),

$$
\begin{aligned}
(\nabla^\alpha\nabla^\beta\varphi)(\tilde{T} \owedge g)_{i\alpha j\beta} &= -\varphi|\operatorname{Hess}\varphi\,|^2 g_{ij} - \varphi(\Delta\varphi)\nabla_i\nabla_j\varphi + 2\varphi g^{\alpha\beta}(\nabla_\alpha\nabla_i\varphi)(\nabla_\beta\nabla_j\varphi) \\
&\quad |\,\mathrm{d}\varphi\,|^2(\Delta\varphi)g_{ij} - |\,\mathrm{d}\varphi\,|^2\nabla_i\nabla_j\varphi \\
&= 2\varphi g^{\alpha\beta}(\nabla_\alpha\nabla_i\varphi)(\nabla_\beta\nabla_j\varphi) - (\varphi\Delta\varphi + |\,\mathrm{d}\varphi\,|^2)\nabla_i\nabla_j\varphi \\
&\quad + \big(|\,\mathrm{d}\varphi\,|^2\Delta\varphi - \varphi|\operatorname{Hess}\varphi\,|^2\big)g_{ij}.
\end{aligned} \tag{4.9}
$$

Finally, (4.6) and (4.7) give

$$
g^{\alpha\beta}(\tilde{T} \owedge g)_{i\alpha j\beta} = -\varphi\nabla_i\nabla_j\varphi - (\varphi\Delta\varphi + 2|\,\mathrm{d}\varphi\,|^2)g_{ij}. \tag{4.10}
$$

Substituting (4.5), (4.8), (4.9) and (4.10) into (4.3) implies

$$\begin{aligned}
\hat{X}_{ij} &= \varphi^2 X_{ij} \\
&= -\frac{1}{2}\varphi\Big(- g^{\alpha\beta}(R_{i\alpha}\nabla_\beta\nabla_j\varphi + R_{\beta j}\nabla_\alpha\nabla_i\varphi) + (\Delta\varphi)R_{ij} \\
&\quad + \frac{1}{2}S\nabla_i\nabla_j\varphi + \Big(R^{\alpha\beta}\nabla_\alpha\nabla_\beta\varphi - \frac{1}{2}S\Delta\varphi\Big)g_{ij}\Big) \\
&\quad + \frac{1}{2}(\varphi\Delta\varphi - |\,\mathrm{d}\varphi\,|^2)R_{ij} \\
&\quad + \frac{1}{2}\Big(\varphi g^{\alpha\beta}(R_{i\alpha}\nabla_\beta\nabla_j\varphi + R_{\beta j}\nabla_\alpha\nabla_i\varphi) - |\,\mathrm{d}\varphi\,|^2 R_{ij} - \frac{1}{2}\varphi S\nabla_i\nabla_j\varphi \\
&\quad + \Big(\frac{1}{2}\varphi S\Delta\varphi - \varphi R^{\alpha\beta}\nabla_\alpha\nabla_\beta\varphi\Big)g_{ij}\Big) \\
&\quad + \frac{1}{2\varphi}\Big(2\varphi g^{\alpha\beta}(\nabla_\alpha\nabla_i\varphi)(\nabla_\beta\nabla_j\varphi) - (\varphi\Delta\varphi + |\,\mathrm{d}\varphi\,|^2)\nabla_i\nabla_j\varphi \\
&\quad + \big(|\,\mathrm{d}\varphi\,|^2\Delta\varphi - \varphi|\,\mathrm{Hess}\,\varphi\,|^2\big)g_{ij}\Big) \\
&\quad - \frac{1}{2\varphi^2}\big(\varphi\Delta\varphi - |\,\mathrm{d}\varphi\,|^2\big)\Big(- \varphi\nabla_i\nabla_j\varphi - (\varphi\Delta\varphi + 2|\,\mathrm{d}\varphi\,|^2)g_{ij}\Big)
\end{aligned}$$

Collecting the terms with $\nabla_i\nabla_j\varphi$ and the terms with g_{ij} gives

$$\begin{aligned}
\hat{X}_{ij} &= \varphi^2 X_{ij} + \varphi g^{\alpha\beta}\big(R_{i\alpha}\nabla_\beta\nabla_j\varphi + R_{\beta j}\nabla_\alpha\nabla_i\varphi\big) \\
&\quad + g^{\alpha\beta}(\nabla_\alpha\nabla_i\varphi)(\nabla_\beta\nabla_j\varphi) - |\,\mathrm{d}\varphi\,|^2 R_{ij} - \Big(\frac{1}{2}\varphi S + \frac{|\,\mathrm{d}\varphi\,|^2}{\varphi}\Big)\nabla_i\nabla_j\varphi \\
&\quad + \Big(- \varphi R^{\alpha\beta}\nabla_\alpha\nabla_\beta\varphi + \frac{1}{2}\varphi S\Delta\varphi - \frac{1}{2}|\,\mathrm{Hess}\,\varphi\,|^2 + \frac{1}{2}(\Delta\varphi)^2 - \frac{|\,\mathrm{d}\varphi\,|^2\Delta\varphi}{\varphi} + \frac{|\,\mathrm{d}\varphi\,|^4}{\varphi^2}\Big)g_{ij}
\end{aligned}$$

as required. □

Motivated by calculations in Chapter 3, it may be useful to understand how the trace and the traceless part of the Cross curvature, given by $X = g^{ij}X_{ij}$ and $\hat{X}^\circ_{ij} = X_{ij} - (X/3)g_{ij}$ respectively, transform under a conformal change of metric.

Theorem 4.6. *Suppose (M^3, g) be a 3 dimensional Riemannian manifold with nonvanishing sectional curvature, $\hat{g} = \frac{1}{\varphi^2}g$, $\varphi \in C^\infty_+(M)$, and $\hat{X}_{ij}$ exists. Then*

$$\begin{aligned}
\hat{X} &= \varphi^2\Big(\varphi^2 X - \varphi R^{\alpha\beta}\nabla_\alpha\nabla_\beta\varphi - S|\,\mathrm{d}\varphi\,|^2 - \frac{1}{2}|\,\mathrm{Hess}\,\varphi\,|^2 + \varphi S\Delta\varphi \\
&\quad + \frac{3}{2}(\Delta\varphi)^2 - 4\frac{|\,\mathrm{d}\varphi\,|^2\Delta\varphi}{\varphi} + 3\frac{|\,\mathrm{d}\varphi\,|^4}{\varphi^2}\Big)
\end{aligned}$$

and

$$\begin{aligned}
\hat{X}^\circ_{ij} &= \varphi^2 X^\circ_{ij} + g^{\alpha\beta}(\nabla_\alpha\nabla_i\varphi)(\nabla_\beta\nabla_j\varphi) + \Big(\frac{1}{6}S\varphi - \frac{|\,\mathrm{d}\varphi\,|^2}{\varphi}\Big)\nabla_i\nabla_j\varphi \\
&\quad + \Big(- \frac{1}{18}S\varphi\Delta\varphi - \frac{1}{3}|\,\mathrm{Hess}\,\varphi\,|^2 + \frac{1}{3}\frac{|\,\mathrm{d}\varphi\,|^2\Delta\varphi}{\varphi}\Big)g_{ij}.
\end{aligned}$$

Proof. Taking the trace of (4.1) with respect to $\hat{g}$ gives

$$\begin{aligned}
\hat{X} &= \varphi^2\Bigg(\varphi^2 X + \varphi g^{ij} g^{\alpha\beta}\big(R_{i\alpha}\nabla_\beta\nabla_j\varphi + g^{ij}R_{\beta j}\nabla_\alpha\nabla_i\varphi\big)\\
&\quad + g^{ij}g^{\alpha\beta}(\nabla_\alpha\nabla_i\varphi)(\nabla_\beta\nabla_j\varphi) - |\,\mathrm{d}\varphi\,|^2 S - \left(\frac{1}{2}\varphi S + \frac{|\,\mathrm{d}\varphi\,|^2}{\varphi}\right)\Delta\varphi\\
&\quad + 3\bigg(-\varphi R^{\alpha\beta}\nabla_\alpha\nabla_\beta\varphi + \frac{1}{2}\varphi S\Delta\varphi - \frac{1}{2}|\operatorname{Hess}\varphi\,|^2 + \frac{1}{2}(\Delta\varphi)^2 - \frac{|\,\mathrm{d}\varphi\,|^2\Delta\varphi}{\varphi} + \frac{|\,\mathrm{d}\varphi\,|^4}{\varphi^2}\bigg)\Bigg).
\end{aligned}$$

By definition, $g^{ij}g^{\alpha\beta}R_{i\alpha} = R^{j\beta}$, so after a change of dummy variables

$$g^{ij}g^{\alpha\beta}R_{i\alpha}\nabla_\beta\nabla_j\varphi = R^{\alpha\beta}\nabla_\alpha\nabla_\beta\varphi.$$

Also, by definition, $g^{ij}g^{\alpha\beta}(\nabla_\alpha\nabla_i\varphi)(\nabla_\beta\nabla_j\varphi) = |\operatorname{Hess}\varphi\,|^2$. Hence,

$$\begin{aligned}
\hat{X} &= \varphi^2\Bigg(\varphi^2 X + 2\varphi R^{\alpha\beta}\nabla_\alpha\nabla_\beta\varphi + |\operatorname{Hess}\varphi\,|^2 - |\,\mathrm{d}\varphi\,|^2 S - \left(\frac{1}{2}\varphi S + \frac{|\,\mathrm{d}\varphi\,|^2}{\varphi}\right)\Delta\varphi\\
&\quad + 3\bigg(-\varphi R^{\alpha\beta}\nabla_\alpha\nabla_\beta\varphi + \frac{1}{2}\varphi S\Delta\varphi - \frac{1}{2}|\operatorname{Hess}\varphi\,|^2 + \frac{1}{2}(\Delta\varphi)^2 - \frac{|\,\mathrm{d}\varphi\,|^2\Delta\varphi}{\varphi} + \frac{|\,\mathrm{d}\varphi\,|^4}{\varphi^2}\bigg)\Bigg)\\
&= \varphi^2\Bigg(\varphi^2 X - \varphi R^{\alpha\beta}\nabla_\alpha\nabla_\beta - |\,\mathrm{d}\varphi\,|^2 S - \frac{1}{2}|\operatorname{Hess}\varphi\,|^2 + S\varphi\Delta\varphi\\
&\quad + \frac{3}{2}(\Delta\varphi)^2 - 4\frac{|\,\mathrm{d}\varphi\,|^2\Delta\varphi}{\varphi} + 3\frac{|\,\mathrm{d}\varphi\,|^4}{\varphi^2}\Bigg).
\end{aligned}$$

Then

$$\begin{aligned}
\hat{X}^\circ_{ij} &= \hat{X}_{ij} - \frac{1}{3}\hat{X}\hat{g}_{ij}\\
&= \varphi^2 X_{ij} + \varphi g^{\alpha\beta}\big(R_{i\alpha}\nabla_\beta\nabla_j\varphi + R_{\beta j}\nabla_\alpha\nabla_i\varphi\big)\\
&\quad + g^{\alpha\beta}(\nabla_\alpha\nabla_i\varphi)(\nabla_\beta\nabla_j\varphi) - |\,\mathrm{d}\varphi\,|^2 R_{ij} - \left(\frac{1}{2}\varphi S + \frac{|\,\mathrm{d}\varphi\,|^2}{\varphi}\right)\nabla_i\nabla_j\varphi\\
&\quad + \bigg(-\varphi R^{\alpha\beta}\nabla_\alpha\nabla_\beta\varphi + \frac{1}{2}\varphi S\Delta\varphi - \frac{1}{2}|\operatorname{Hess}\varphi\,|^2 + \frac{1}{2}(\Delta\varphi)^2 - \frac{|\,\mathrm{d}\varphi\,|^2\Delta\varphi}{\varphi} + \frac{|\,\mathrm{d}\varphi\,|^4}{\varphi^2}\bigg)g_{ij}\\
&\quad - \frac{1}{3}\varphi^2\Bigg(\varphi^2 X - \varphi R^{\alpha\beta}\nabla_\alpha\nabla_\beta\varphi - S|\,\mathrm{d}\varphi\,|^2 - \frac{1}{2}|\operatorname{Hess}\varphi\,|^2 + \varphi S\Delta\varphi\\
&\quad + \frac{3}{2}(\Delta\varphi)^2 - 4\frac{|\,\mathrm{d}\varphi\,|^2\Delta\varphi}{\varphi} + 3\frac{|\,\mathrm{d}\varphi\,|^4}{\varphi^2}\Bigg)\cdot\frac{1}{\varphi^2}g_{ij}\\
&= \varphi^2 X^\circ_{ij} + \varphi g^{\alpha\beta}\big(R_{i\alpha}\nabla_\beta\nabla_j\varphi + R_{\beta j}\nabla_\alpha\nabla_i\varphi\big)\\
&\quad + g^{\alpha\beta}(\nabla_\alpha\nabla_i\varphi)(\nabla_\beta\nabla_j\varphi) - |\,\mathrm{d}\varphi\,|^2 R_{ij} - \left(\frac{1}{2}\varphi S + \frac{|\,\mathrm{d}\varphi\,|^2}{\varphi}\right)\nabla_i\nabla_j\varphi\\
&\quad + \left(-\frac{1}{18}S\varphi\Delta\varphi - \frac{1}{3}|\operatorname{Hess}\varphi\,|^2 + \frac{1}{3}\frac{|\,\mathrm{d}\varphi\,|^2\Delta\varphi}{\varphi}\right)g_{ij}.
\end{aligned}$$

□

4.2.1 Einstein Manifold

Since the Cross Curvature is the adjugate of the Einstein tensor, it is natural to consider the case that (M, g) is an Einstein manifold. Recall (M^n, g), $n \geqslant 3$, is an Einstein manifold if

$$\operatorname{Ric} g = \lambda g$$

for some real number λ. By taking the trace with respect to g, one can see that $S = \lambda n$. This assumption leads to an explicit formula for X_{ij} in terms of S and g_{ij}. Moreover, the formulas presented in Theorems 4.5 and 4.6 can be simplified to only depend on X_{ij}, derivatives of φ, S and g_{ij}.

Proposition 4.7. *Suppose (M^n, g), $n \geqslant 3$, be an Einstein manifold. Then X_{ij} exists if and only if $S \neq 0$ and if $S \neq 0$ then*

$$X_{ij} = \left(\frac{1}{n} - \frac{1}{2}\right)^{n-1} S^{n-1} g_{ij}.$$

In particular, if $n = 3$,

$$X_{ij} = \frac{1}{36} S^2 g_{ij}.$$

Proof. Since $R_{ij} = (1/n) S g_{ij}$ it follows that the Einstein tensor is given by

$$E_{ij} = \left(\frac{1}{n} - \frac{1}{2}\right) S g_{ij}.$$

It follows that

$$\det E = \left(\frac{1}{n} - \frac{1}{2}\right)^n S^n.$$

Since $n \geqslant 3$ implies $1/n - 1/2 \neq 0$, E_{ij} is invertible if and only if $S \neq 0$ and, in the case $S \neq 0$, the inverse of $E^{ij} = g^{i\alpha} g^{j\beta} E_{\alpha\beta}$ is given by

$$V_{ij} = \left(\frac{1}{n} - \frac{1}{2}\right)^{-1} S^{-1} g_{ij}.$$

Using that $X_{ij} = (\det E) V_{ij}$ gives the result. $\square$

Theorem 4.8. *Suppose (M^3, g) be an Einstein manifold with $S \neq 0$, $\hat{g} = \frac{1}{\varphi^2} g$, $\varphi \in C_+^\infty(M)$, and $\hat{X}_{ij}$ exists. Then*

$$\begin{aligned} \hat{X}_{ij} = \varphi^2 X_{ij} + g^{\alpha\beta}(\nabla_i \nabla_\alpha \varphi)(\nabla_\beta \nabla_j \varphi) + \left(\frac{1}{6} S \varphi \Delta\varphi - \frac{|\,\mathrm{d}\varphi\,|^2}{\varphi}\right) \nabla_i \nabla_j \varphi \\ + \left(\frac{1}{6} S\varphi\Delta\varphi - \frac{1}{2} |\operatorname{Hess}\varphi\,|^2 + \frac{1}{2}(\Delta\varphi)^2 - \frac{|\,\mathrm{d}\varphi\,|^2 \Delta\varphi}{\varphi} + \frac{|\,\mathrm{d}\varphi\,|^4}{\varphi^2}\right) g_{ij}. \end{aligned}$$

4.3 The Realisability Conjecture for Cross Curvature

The realizability conjecture for cross curvature is a conjecture made by Richard Hamilton regarding uniquely prescribing the so called $(1,2)$ Cross Curvature on S^3. In this section we will prove the uniqueness part of the statement in a conformal class. Before we do that or, for that matter, state the conjecture, we need to state some definitions and small results. Suppose (M^3, g) is a Riemannian manifold and let ϵ_{ijk} be the Levi-Cevita symbol defined by

$$\epsilon_{ijk} = \begin{cases} 1, & \text{if } (i,j,k) \text{ is an even permutation of } (1,2,3) \\ -1, & \text{if } (i,j,k) \text{ is an odd permutation of } (1,2,3) \\ 0 & \text{else} \end{cases}$$

where the even permutations of (1,2,3) are (1,2,3), (2,3,1), (3,1,2) and the odd permutations are (3,2,1), (2,1,3), (1,3,2).

Definition. The *(1,2)-Cross Curvature* is given by

$$X_{ij}^k = \frac{1}{2}\epsilon^{\alpha\beta k}R_{\alpha\beta ij}$$

where $\epsilon^{ijk} = g^{\alpha i}g^{\beta j}g^{\gamma k}\epsilon_{\alpha\beta\gamma}$.

Then the Realisability Conjecture states every positive cross tensor on $\mathbf{S}^3$ is the (1,2) Cross Curvature of a unique metric. A *cross tensor* is a (1,2)-tensor T on an orientated Riemannain manifold (M^3, g) such that

1. $T_{ij}^k = -T_{ij}^k$ for all $i, j, k = 1, 2, 3$ and
2. $T_{i\alpha}^{\alpha} = 0$ for all $i = 1, 2, 3$.

A cross tensor which also satisfies

3 For all $p \in M$ if $u, v \in T_pM$ are two linearly independent vectors then $\{u, v, \mathrm{T}(u, v)\}$ forms a positively orientated basis of T_pM.

is called a *cross tensor*. One can easily check that (1,2)-Cross Curvature is a positive cross tensor on $\mathbf{S}^3$. Indeed, the (1,2)-cross curvature satisfies 1. and 2. on any 3 dimensional Riemannain manifold. It can also be shown that if (M^3, g) is orientated then X_{ij}^k is positive if and only if (M, g) has positive sectional curvature, see [14]. This is related to a realizability conjecture made by R. Hamilton regarding existence and uniqueness of metrics on $\mathbf{S}^3$ with prescribed cross curvature.

The main result in this section is the following.

Theorem 4.9. *Suppose* (M^3, g) *is an Einstein manifold with positive sectional curvature and* $\hat{g} = \frac{1}{\varphi^2} g$, $\varphi \in C^\infty_+(M)$. *If* $X^k_{ij} = \hat{X}^k_{ij}$ *then* $\varphi = 1$.

Theorem 4.9 proves that on an orientated Einstein spaces there is at most one metric in a conformal class which realises a given positive cross tensor. In particular, we have the following corollary

Corollary 4.10. *For every positive cross tensor* T *on* S^3 *there is at most one metric in a conformal class with* T *as its (1,2) Cross Curvature.*

In fact, Theorem 4.9 follows from the slightly more general result:

Theorem 4.11. *Suppose* (M^n, g) *is an Einstein manifold with positive sectional curvature and* $\hat{g} = \frac{1}{\varphi^2} g$, $\varphi \in C^\infty_+(M)$. *If* $X_{ij} = \hat{X}_{ij}$ *then* $\varphi = 1$.

One can see that Theorem 4.9 follows from Theorem 4.11 by virtue of the following proposition.

Proposition 4.12. *Suppose* (M^3, g) *has nonvanishing sectional curvature. Then*

$$X_{ij} = \frac{1}{2} X^\alpha_{i\beta} X^\beta_{\alpha j}.$$

Proof. Fix a point $p \in M$ and choose coordinates such that E_{ij} is diagonal and $g_{ij} = \delta_{ij}$ at p as in the proof of Proposition 4.1. Recall the only independent, nonzero terms in Riemannian curvature are the sectional curvatures

$$a := R_{1212}, \quad b := R_{1313} \quad c := R_{2323}$$

and the Cross Curvature is given by

$$X_{ij} = \begin{pmatrix} ab & 0 & 0 \\ 0 & ac & 0 \\ 0 & 0 & bc \end{pmatrix}.$$

Due to the antisymmetry of ϵ^{ijk}, we have

$$\begin{aligned}
X^1_{ij} &= \frac{1}{2} \epsilon^{\alpha\beta 1} R_{\alpha\beta ij} \\
&= \frac{1}{2} \big(R_{23ij} - R_{32ij} \big) \\
&= \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & c \\ 0 & -c & 0 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -c \\ 0 & c & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & c \\ 0 & -c & 0 \end{pmatrix}.
\end{aligned}$$

Similarly,

$$X_{ij}^2 = \begin{pmatrix} 0 & 0 & -b \\ 0 & 0 & 0 \\ b & 0 & 0 \end{pmatrix} \text{ and } X_{ij}^3 = \begin{pmatrix} 0 & a & 0 \\ -a & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Now we wish to calculate $X_{i\beta}^{\alpha} X_{\alpha j}^{\beta}$.Observe that for all i, $X_{i1}^1 = X_{i2}^2 = X_{i3}^3$. Moreover, since $X_{ij}^k = -X_{ji}^k$, $X_{i\beta}^{\alpha} X_{\alpha j}^{\beta} = X_{j\beta}^{\alpha} X_{\alpha i}^{\beta}$. This implies that

$$X_{i\beta}^{\alpha} X_{\alpha j}^{\beta} = 2(X_{i2}^1 X_{1j}^2 + X_{i3}^1 X_{1j}^3 + X_{i3}^2 X_{2j}^3).$$

By the above calculations, it follows

$$\begin{aligned} X_{i\beta}^{\alpha} X_{\alpha j}^{\beta} &= 2(-c\delta_{i3})(-b\delta_{3j}) + 2(c\delta_{i2})(a\delta_{2j}) + 2(-b\delta_{i1})(-a\delta_{1j}) \\ &= 2 \begin{pmatrix} ab & 0 & 0 \\ 0 & ac & 0 \\ 0 & 0 & bc \end{pmatrix} \\ &= 2X_{ij}. \end{aligned}$$

□

Before we prove Theorem 4.11, we require the following Lemma.

Lemma 4.13. *Suppose (M^n, g) has positive sectional curvature. Then $X_{ij} = \lambda g_{ij}$ for some $\lambda \neq 0$ if and only if (M, g) is Einstein.*

Proof. We have already shown in Proposition 4.7 that if (M, g) is Einstein then $X_{ij} = \lambda g_{ij}$ where $\lambda = \frac{1}{36} S^2$. The scalar curvature is nonzero (in fact positive) since (M, g) has positive sectional curvature. Now suppose $X_{ij} = \lambda g_{ij}$. Then $V_{ij} = \frac{\lambda}{\det E} g_{ij}$, so $E^{ij} = \frac{\det E}{\lambda} g^{ij}$. It follows that

$$E_{ij} = \frac{\det E}{\lambda} g_{ij}$$

which implies that (M, g) is Einstein.

□

Proof of Theorem 4.11. Since (M, g) is Einstein, Lemma 4.13 implies there exists $\lambda \neq 0$ such that $X_{ij} = \lambda g_{ij}$. This implies

$$\hat{X}_{ij} = \lambda g_{ij} = \lambda \varphi^2 \hat{g}_{ij}.$$

Using Lemma 4.13 again implies $(M, \hat{g})$ is Einstein. It follows from Proposition 4.7 that $\hat{X}_{ij} = \frac{1}{36} \hat{S}^2 g_{ij}$ which implies φ is a constant (it must equal to $\frac{\hat{S}}{6\sqrt{\lambda}}$). But if φ is constant it follows from Theorem 4.5 (or a simple calculation) that

$$X_{ij} = \hat{X}_{ij} = \varphi^2 X_{ij}.$$

Thus, $\varphi = 1$.

□

Appendix A

Riemannian Curvature under a Conformal Transformation

Let $\hat{g}$ is conformal to g. If Ω is a quantity formed with respect to g then the corresponding quantity formed with respect to $\hat{g}$ will be denoted by $\hat{\Omega}$. In order to find how the Riemannian Curvature changes under a conformal transformation, we must first consider how the Levi-Cevita connection changes under such a transformation.

Theorem A.1. *Suppose $\hat{g} = e^{2\rho} g$, $\rho \in C^\infty(M)$. For all $X, Y \in \mathfrak{X}(M)$,*

$$\hat{\nabla}_X Y = \nabla_X Y + \mathrm{d}\varphi(X) Y + \mathrm{d}\rho(Y) X - g(X, Y) \operatorname{grad} \rho$$

Proof. Fix a set of local coordinates $\{x^i\}_{i=1}^n$. The Christoffel symbols are given by

$$\Gamma^k_{ij} = \frac{1}{2} g^{ks} \left(\frac{\partial g_{is}}{\partial x^j} + \frac{\partial g_{js}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^s} \right).$$

Observe that

$$\frac{\partial \hat{g}_{ij}}{\partial x^s} = 2e^{2\rho} g_{ij} \nabla_s \rho + e^{2\rho} \frac{\partial g_{ij}}{\partial x^s}$$

so

$$\begin{aligned}
\hat{\Gamma}^k_{ij} &= \frac{1}{2} e^{-2\rho} g^{ks} \bigg(e^{2\rho} \frac{\partial g_{is}}{\partial x^j} + e^{2\rho} \frac{\partial g_{js}}{\partial x^i} - e^{2\rho} \frac{\partial g_{ij}}{\partial x^s} \\
&\qquad + 2e^{2\rho} g_{is} \nabla_j \rho + 2e^{2\rho} g_{js} \nabla_i \rho - 2e^{2\rho} g_{ij} \nabla_s \rho \bigg) \\
&= \Gamma^k_{ij} + \delta^k_i \nabla_j \rho + \delta^k_j \nabla_i \rho - g_{ij} \nabla^k \rho
\end{aligned}$$

as required. □

Definition. The *Kulkarni-Nomizu product*, denoted $\owedge$, maps two symmetric $(0, 2)$-tenors a and b to the (0,4)-tensor $a \owedge b$ defined by

$$\begin{aligned}
(a \owedge b)(X, Y, Z, W) &= a(X, Z) b(Y, W) + a(Y, W) b(X, Z) \\
&\qquad - a(X, W) b(Y, Z) - a(Y, Z) b(X, W).
\end{aligned}$$

Theorem A.2. *Suppose $\hat{g} = \frac{1}{\varphi^2} g$, $\varphi \in C^\infty_+(M)$. There holds*

$$\operatorname{Rm} \hat{g} = \frac{1}{\varphi^2} \operatorname{Rm} g - \frac{1}{\varphi^4} g \owedge \Big(-\varphi \operatorname{Hess} \varphi + \frac{1}{2} |\, \mathrm{d}\varphi \,|^2 g \Big)$$

Proof. In local coordinates, the (1,3)-Riemannin curvature tensor is given by

$$R_{ijk}{}^s = \frac{\partial \Gamma^s_{ik}}{\partial x^j} - \frac{\partial \Gamma^s_{jk}}{\partial x^i} + \Gamma^t_{ik}\Gamma^s_{tj} - \Gamma^t_{jk}\Gamma^s_{ti}.$$

It follows that

$$\begin{aligned} \hat{R}_{ijkl} &= \hat{g}_{ls} \left(\frac{\partial \hat{\Gamma}^s_{ik}}{\partial x^j} - \frac{\partial \hat{\Gamma}^s_{jk}}{\partial x^i} + \hat{\Gamma}^t_{ik}\hat{\Gamma}^s_{tj} - \hat{\Gamma}^t_{jk}\hat{\Gamma}^s_{ti} \right) \\ &= e^{2\rho} g_{ls} \left(\frac{\partial \hat{\Gamma}^s_{ik}}{\partial x^j} - \frac{\partial \hat{\Gamma}^s_{jk}}{\partial x^i} \right) + e^{2\rho} g_{ls} \big(\hat{\Gamma}^t_{ik}\hat{\Gamma}^s_{tj} - \hat{\Gamma}^t_{jk}\hat{\Gamma}^s_{ti} \big). \end{aligned}$$

Firstly, by Theorem A.1,

$$\frac{\partial \hat{\Gamma}^s_{ik}}{\partial x^j} = \frac{\partial \Gamma^s_{ik}}{\partial x^j} + \delta^s_i \frac{\partial}{\partial x^j}\big(\nabla_k \rho\big) + \delta^s_k \frac{\partial}{\partial x^j}\big(\nabla_i \rho\big) - \frac{\partial}{\partial x^j}\big(g_{ik}\nabla^s \rho\big).$$

Then,

$$\nabla_j \nabla_k \rho = \frac{\partial}{\partial x^j}\big(\nabla_k \rho\big) - \Gamma^t_{jk}\nabla_t \rho$$

and

$$\nabla_j \big(g_{ik}\nabla^s \rho\big) = \frac{\partial}{\partial x^j}\big(g_{ik}\nabla^s \rho\big) - g_{tk}\Gamma^t_{ij}\nabla^s \rho - g_{it}\Gamma^t_{jk}\nabla^s \rho + g_{ik}\Gamma^s_{tj}\nabla^t \rho.$$

Hence,

$$\begin{aligned} \frac{\partial \hat{\Gamma}^s_{ik}}{\partial x^j} = \frac{\partial \Gamma^s_{ik}}{\partial x^j} &+ \delta^s_i \big(\nabla_j \nabla_k \rho + \Gamma^t_{jk}\nabla_t \rho\big) + \delta^s_k \big(\nabla_i \nabla_j \rho + \Gamma^t_{ij}\nabla_t \rho\big) \\ &- g_{ik}\nabla_j \nabla^s \rho - \big(g_{kt}\Gamma^t_{ij} + g_{it}\Gamma^t_{jk}\big)\nabla^s \rho + g_{ik}\Gamma^s_{tj}\nabla^t \rho. \end{aligned}$$

The terms $\nabla_i \nabla_j \rho + \Gamma^t_{ij}\nabla_t \rho$ and $g_{kt}\Gamma^t_{ij}\nabla^s$ are symmetric in i and j, which means they will vanish in the expression for $\frac{\partial \hat{\Gamma}^s_{ik}}{\partial x^j} - \frac{\partial \hat{\Gamma}^s_{jk}}{\partial x^i}$. It follows that

$$\begin{aligned} \frac{\partial \hat{\Gamma}^s_{ik}}{\partial x^j} - \frac{\partial \hat{\Gamma}^s_{jk}}{\partial x^i} = \frac{\partial \Gamma^s_{ik}}{\partial x^j} - \frac{\partial \Gamma^s_{jk}}{\partial x^i} &+ \delta^s_i \big(\nabla_j \nabla_k \rho + \Gamma^t_{jk}\nabla_t \rho\big) - \delta^s_j \big(\nabla_i \nabla_k \rho + \Gamma^t_{ik}\nabla_t \rho\big) \\ &- \big(g_{ik}\nabla_j \nabla^s \rho - g_{jk}\nabla_i \nabla^s \rho\big) - \big(g_{it}\Gamma^t_{jk} - g_{jt}\Gamma^t_{ik}\big)\nabla^s \rho \\ &+ \big(g_{ik}\Gamma^s_{tj} - g_{jk}\Gamma^s_{ti}\big)\nabla^t \rho. \end{aligned}$$

Contracting gives

$$\begin{aligned} \hat{g}_{sl}\left(\frac{\partial \hat{\Gamma}^s_{ik}}{\partial x^j} - \frac{\partial \hat{\Gamma}^s_{jk}}{\partial x^i} \right) = e^{2\rho} g_{sl} \left(\frac{\partial \Gamma^s_{ik}}{\partial x^j} - \frac{\partial \Gamma^s_{jk}}{\partial x^i} \right) &- e^{2\rho}\Big(g_{ik}\big(\nabla_j \nabla_l \rho - g_{ls}\Gamma^s_{tj}\nabla^t \rho\big) \\ &- g_{il}\big(\nabla_j \nabla_k \rho + \Gamma^t_{jk}\nabla_t \rho\big) + g_{jl}\big(\nabla_i \nabla_k \rho + \Gamma^t_{ik}\nabla_t \rho\big) \\ &- g_{jk}\big(\nabla_i \nabla_l \rho - g_{ls}\Gamma^s_{ti}\nabla^t \rho\big) + \big(g_{it}\Gamma^t_{jk} - g_{jt}\Gamma^t_{ik}\big)\nabla_l \rho \Big). \end{aligned} \tag{A.1}$$

Secondly, again by Theorem A.1, we have

$$
\begin{aligned}
\hat{g}_{ls}\hat{\Gamma}^t_{ik}\hat{\Gamma}^s_{tj} &= e^{2\rho}\big(\Gamma^t_{ik} + \delta^t_k\nabla_i\rho + \delta^t_i\nabla_k\rho - g_{ik}\nabla^t\rho\big) \\
&\qquad \big(g_{ls}\Gamma^s_{tj} + g_{lt}\nabla_j\rho + g_{jl}\nabla_t\rho - g_{tj}\nabla_l\rho\big) \\
&= e^{2\rho}\big(g_{ls}\Gamma^t_{ik}\Gamma^s_{tj} + g_{lt}\Gamma^t_{ik}\nabla_j\rho + g_{jl}\Gamma^t_{ik}\nabla_t\rho - g_{tj}\Gamma^t_{ik}\nabla_l\rho \\
&\qquad + g_{ls}\Gamma^s_{jk}\nabla_i\rho + g_{kl}\nabla_i\rho\nabla_j\rho + g_{jl}\nabla_i\rho\nabla_k\rho - g_{jk}\nabla_i\rho\nabla_l\rho \\
&\qquad + g_{ls}\Gamma^s_{ij}\nabla_k\rho + g_{il}\nabla_k\rho\nabla_j\rho + g_{jl}\nabla_i\rho\nabla_k\rho - g_{ij}\nabla_k\rho\nabla_l\rho \\
&\qquad - g_{ik}g_{ls}\Gamma^s_{tj}\nabla^t\rho - g_{ik}\nabla_j\rho\nabla_l\rho - g_{jl}g_{ik}|\,\mathrm{d}\rho\,|^2 + g_{ik}\nabla_j\rho\nabla_l\rho\big) \\
&= e^{2\rho}\big(g_{ls}\Gamma^t_{ik}\Gamma^s_{tj} + g_{lt}\Gamma^t_{ik}\nabla_j\rho + g_{jl}\Gamma^t_{ik}\nabla_t\rho - g_{tj}\Gamma^t_{ik}\nabla_l\rho \\
&\qquad + g_{ls}\Gamma^s_{jk}\nabla_i\rho + g_{kl}\nabla_i\rho\nabla_j\rho + g_{jl}\nabla_i\rho\nabla_k\rho - g_{jk}\nabla_i\rho\nabla_l\rho \\
&\qquad + g_{ls}\Gamma^s_{ij}\nabla_k\rho + g_{il}\nabla_k\rho\nabla_j\rho + g_{jl}\nabla_i\rho\nabla_k\rho - g_{ij}\nabla_k\rho\nabla_l\rho \\
&\qquad - g_{ik}g_{ls}\Gamma^s_{tj}\nabla^t\rho - g_{jl}g_{ik}|\,\mathrm{d}\rho\,|^2\big).
\end{aligned}
$$

As before, the terms: $g_{kl}\nabla_i\rho\nabla_j\rho$; $g_{ls}\Gamma^s_{ij}\nabla_k\rho$; $g_{ij}\nabla_k\rho\nabla_l\rho$; $g_{jl}\nabla_i\rho\nabla_k\rho + g_{il}\nabla_j\rho\nabla_k\rho$; and $g_{lt}\Gamma^t_{ik}\nabla_j\rho + g_{ls}\Gamma^s_{jk}\nabla_i\rho$ are symmetric in i and j, so they will not appear in the expression for $\hat{g}_{ls}\hat{\Gamma}^t_{ik}\hat{\Gamma}^s_{tj} - \hat{g}_{ls}\hat{\Gamma}^t_{jk}\hat{\Gamma}^s_{ti}$. Hence,

$$
\begin{aligned}
\hat{g}_{ls}\hat{\Gamma}^t_{ik}\hat{\Gamma}^s_{tj} - \hat{g}_{ls}\hat{\Gamma}^t_{jk}\hat{\Gamma}^s_{ti} &= e^{2\rho}\big(g_{ls}\Gamma^t_{ik}\Gamma^s_{tj} - g_{ls}\Gamma^t_{jk}\Gamma^s_{ti} + \big(g_{jl}\Gamma^t_{ik} - g_{il}\Gamma^t_{jk}\big)\nabla_t\rho \\
&\qquad - \big(g_{jt}\Gamma^t_{ik} - g_{it}\Gamma^t_{jk}\big)\nabla_l\rho - \big(g_{jk}\nabla_i\rho - g_{ik}\nabla_j\rho\big)\nabla_l\rho \\
&\qquad + (g_{jl}\nabla_i\rho - g_{il}\nabla_j\rho)\nabla_k\rho - \big(g_{ik}g_{ls}\Gamma^s_{tj} - g_{jk}g_{ls}\Gamma^s_{ti}\big)\nabla^t\rho \\
&\qquad - \big(g_{jl}g_{ik} - g_{il}g_{jk}\big)|\,\mathrm{d}\rho\,|^2\big) \\
&= e^{2\rho}\big(g_{ls}\Gamma^t_{ik}\Gamma^s_{tj} - g_{ls}\Gamma^t_{jk}\Gamma^s_{ti}\big) \\
&\qquad - e^{2\rho}\bigg(g_{ik}\bigg(- \nabla_j\rho\nabla_l\rho + \frac{1}{2}|\,\mathrm{d}\rho\,|^2 g_{jl} + g_{ls}\Gamma^s_{tj}\nabla^t\rho\bigg) \\
&\qquad - g_{il}\bigg(- \nabla_j\rho\nabla_k\rho + \frac{1}{2}|\,\mathrm{d}\rho\,|^2 g_{jk} - \Gamma^t_{jk}\nabla_t\rho\bigg) \\
&\qquad + g_{jl}\bigg(- \nabla_i\rho\nabla_k\rho + \frac{1}{2}|\,\mathrm{d}\rho\,|^2 g_{ik} - \Gamma^t_{ik}\nabla_t\rho\bigg) \\
&\qquad - g_{jk}\bigg(- \nabla_i\rho\nabla_l\rho + \frac{1}{2}|\,\mathrm{d}\rho\,|^2 g_{il}\rho + g_{ls}\Gamma^s_{ti}\nabla^t\rho\bigg) \\
&\qquad + \big(g_{it}\Gamma^t_{jk} - g_{jt}\Gamma^t_{ik}\big)\nabla_l\rho\bigg). \qquad \text{(A.2)}
\end{aligned}
$$

Thus, combining (A.1) and (A.2) gives

$$\begin{aligned}\hat{R}_{ijkl} = e^{2\rho} R_{ijkl} - e^{2\rho}\bigg(g_{ik}\bigg(\nabla_j \nabla_l \rho - \nabla_j \rho \nabla_l \rho + \frac{1}{2} |\,\mathrm{d}\rho\,|^2 g_{jl}\bigg) \\ - g_{il}\bigg(\nabla_j \nabla_k \rho - \nabla_j \rho \nabla_k \rho + \frac{1}{2} |\,\mathrm{d}\rho\,|^2 g_{jk}\bigg) \\ + g_{jl}\bigg(\nabla_i \nabla_k \rho - \nabla_i \rho \nabla_k \rho + \frac{1}{2} |\,\mathrm{d}\rho\,|^2 g_{ik}\bigg)\bigg) \\ - g_{jk}\bigg(\nabla_i \nabla_l \rho - \nabla_i \rho \nabla_l \rho + \frac{1}{2} |\,\mathrm{d}\rho\,|^2 g_{il}\bigg).\end{aligned}$$

Finally, $\rho = -\log\varphi$, so

$$\nabla_i \nabla_j \rho - \nabla_i \rho \nabla_j \rho + \frac{1}{2} |\,\mathrm{d}\rho\,|^2 g_{ij} = \frac{1}{\varphi^2}\bigg(-\varphi \nabla_i \nabla_j \varphi + \frac{1}{2} |\,\mathrm{d}\varphi\,|^2 g_{ij}\bigg),$$

so

$$\hat{R}_{ijkl} = \frac{1}{\varphi^2} R_{ijkl} - \frac{1}{\varphi^4} (g \owedge T)_{ijkl}$$

where $T_{ij} = -\varphi \nabla_i \nabla_j \varphi + \frac{1}{2} |\,\mathrm{d}\varphi\,|^2 g_{ij}$.

□

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